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New correlation functions for random matrices and integrals over supergroups

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Abstract
The averages of ratios of characteristic polynomials $\det(\lambda - X)$ of $N \times N$ random matrices X are investigated in the large-N limit for the GUE, GOE and GSE ensembles. The density of states and the two-point correlation function are derived from these ratios. The method relies on an extension of the Harish-Chandra, Itzykson, Zuber integrals to the GOE ensemble and to supergroups, which are explicitly computed through heat kernel differential equations. An external matrix source, linearly coupled to the random matrices, may also be added to the Gaussian distribution, and allows for a discussion of the universality of the GOE results in the large-N limit.

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1. Introduction

In this paper we consider correlation functions involving characteristic polynomials of $N \times N$ random matrices X belonging to the Gaussian unitary ensemble (GUE), Gaussian orthogonal ensemble (GOE) or Gaussian symplectic ensemble (GSE) of the following type

$$F(\lambda_1, \ldots, \lambda_k; \mu_1 \ldots \mu_k) = \left\langle \prod_{\alpha=1}^{k} \frac{\det(\lambda_\alpha - X)}{\det(\mu_\alpha - X)} \right\rangle. \tag{1}$$

The reason for considering expectation values of such characteristic polynomials is as follows. First, they turn out to be simpler than the usual correlation functions of the type $\left\langle \prod_{\alpha=1}^{k} \mathrm{Tr} \frac{1}{z_\alpha - X} \right\rangle$, but they may be used to recover the same information. For instance

$$\frac{\partial}{\partial\lambda} \left\langle \frac{\det(\lambda - X)}{\det(\mu - X)} \right\rangle \Bigg|_{\mu=\lambda} = \left\langle \mathrm{Tr} \frac{1}{\lambda - X} \right\rangle \tag{2}$$

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and so on. Next we know [4–6] that correlations of products $\left\langle \prod_{\alpha=1}^{k} \det(\lambda_\alpha - X) \right\rangle$ are universal in the Dyson limit, in which the distance between the λ is of the order of the mean spacing [1–3]. This universality is known to hold if the probability distribution is non-Gaussian, or if it includes an external matrix source linearly coupled to the random matrices. Presumably, this extends to the ratios (1) considered here and some evidence in that direction will be presented below, when the Gaussian measure is modified by the presence of an external source.

The calculations will be shown to involve integrals over groups which often go beyond the case considered by Harish-Chandra, Itzykson and Zuber (HIZ) [9, 10], and the main point of this paper is to analyse such cases. Such integrals appear at many places in random matrix theory. For instance, they are essential in the problem of a random Hamiltonian H, a Hermitian $N \times N$ matrix, which is the sum of a non-random H_0 and a Gaussian random V:

$$H = H_0 + V \tag{3}$$

with

$$P(V) = \frac{1}{Z} \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\, V^2}. \tag{4}$$

The probability law for the matrix elements of H is thus

$$P(H) = \frac{1}{Z'} \mathrm{e}^{-\frac{N}{2}\mathrm{Tr} H^2 + N \mathrm{Tr}\, H H_0}. \tag{5}$$

When H_0 is non-zero this measure is not invariant under a change of basis, i.e. under the orthogonal, unitary or symplectic groups appropriate to real symmetric, Hermitian or quaternionic matrices. If one wants to write now the probability distribution for the eigenvalues of H, one obtains $H = U \Lambda U^{-1}$ with Λ diagonal, and U a group element appropriate to the ensemble. This yields a well-known Jacobian

$$\mathrm{d}H = \mathrm{d}U |\Delta(\lambda_1, \ldots, \lambda_N)|^\beta \prod_1^N \mathrm{d}\lambda_a \tag{6}$$

with $\beta = 1, 2$ or 4, and the probability distribution for the eigenvalues is given by

$$P(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} |\Delta(\lambda_1, \ldots, \lambda_N)|^\beta \, \mathrm{e}^{-\frac{N}{2}\sum \lambda_a^2} \int \mathrm{d}U \, \mathrm{e}^{N \mathrm{Tr}(U \Lambda U^{-1} H_0)}. \tag{7}$$

Therefore a group integration has to be performed.

Similar integrals occur in many places in random matrix theory. For instance, for chains of random matrices on a lattice with couplings of the form $\exp \mathrm{Tr}\, H_n H_{n+1}$, clearly the joint probability distribution for the eigenvalues involves a similar group integration over the relative group element $U_n^{-1} U_{n+1}$.

In the present work, the correlation functions (1) will be written as supersymmetric integrals and diagonalization through supergroup transformations will lead us to consider new types of HIZ group integrals.

The literature on supersymmetric integration is very rich. Efetov [8] has already obtained expressions of the correlations for those same ratios (1) by supersymmetric techniques. In the unitary case, in which X is a complex Hermitian matrix, the calculation of the k-point correlation is done explicitly with the help of the HIZ integral, which has been extended also to the supermatrix case [11, 12]. However for real symmetric matrices, for general k, the supermatrix formulation becomes complicated since the relevant HIZ integrals are no longer explicit and simple. Recently, Guhr and Kohler [14, 15] have studied those integrals by a recursion formula involving the size N of the matrix. It is a different approach which is proposed below. In a previous work, we have considered such integrals for the symplectic group [7], with the help of differential equations satisfied by the heat kernel for motions in the orthogonal group. This approach may be extended to include an external matrix source, a modification for which the standard method based on orthogonal polynomials does not apply.

Since we have considered such HIZ integrals in the past, let us summarize what was known earlier, beyond the standard HIZ integral for the GUE. In the GOE case a mapping of the $N \times N$ integral for $\langle \prod_1^k \det(\lambda_\alpha - X) \rangle$ to an integral over $k \times k$ matrices invariant over the symplectic group was derived in [7]. Then one found, through the heat kernel differential equation, that the HIZ integral was exactly given in that case by a semiclassical approximation corrected by a *finite* number of terms. Those additional terms are important, since they are of order one in the Dyson limit; once those terms are known one can determine this limit by a saddle-point method. Here we return to those heat kernel equations for the supergroups relevant to our present problem. It turns out that now the series of correction terms to the semiclassical limit does not terminate. However, in the scaling limit this series may be determined explicitly, and the problem at hand is thus solved.

The set-up of this paper is as follows. We first consider the average of a single ratio $\langle \frac{\det(\lambda - X)}{\det(\mu - X)} \rangle$ for the GUE and reduce it directly to quadratures, or alternatively we make use of supersymmetric techniques with a simple version of the HIZ integral. Then the same is done for the GOE. For a single ratio the integration over the supergroup variables is done through a generalization of the HIZ formula, which is exactly given by the semiclassical approximation and a finite number of corrections. We then apply those supersymmetric techniques to higher correlation functions. The required HIZ formula for supergroup integration leads to a differential equation whose solution shows that the semiclassical integration has now to be corrected by an infinite series. This infinite series originates from the part of the supergroup integration which comes from the orthogonal subgroup, and each term of this series is of order one in the Dyson limit. However, in the limit of large matrices it turns out that one need only consider a special case of those HIZ integrals, when the source matrices have only two distinct eigenvalues. Then the whole series may be found in a closed form and one obtains explicitly the large-N limit of the correlation functions for ratios of characteristic polynomials. This is then generalized to k-point functions and, continuing in k, one can study the zero-replica limit $k \to 0$, which is an alternative way of deriving the usual correlation functions from those determinental expectations. The Gaussian symplectic ensemble (GSE) is discussed within the same approach. Those techniques are then extended further to probability measures involving an external matrix source. This allows one to verify that the results in the Dyson limit are not modified by this source. Level spacing distributions are then derived within the same methods. Finally edge singularities are discussed as well.

2. Single ratio of characteristic polynomials in the GUE ensemble

Let us first consider a single ratio of characteristic polynomials, defined as

$$F_N(\lambda, \mu) = \left\langle \frac{\det(\lambda - X)}{\det(\mu - X)} \right\rangle. \tag{8}$$

The averages $\langle \cdots \rangle$ are computed with respect to the Gaussian distribution

$$P(X) = \frac{1}{Z} \exp\left(-\frac{N}{2} \operatorname{Tr} X^2 \right) \tag{9}$$

where the matrix X is an $N \times N$ complex Hermitian matrix.

The determinant $\det(\lambda - X)$ may be expressed as an integral over Grassmann variables $\bar{\theta}$ and θ [7]. The determinant $\det(\mu - X)$ in the denominator may be expressed as a Gaussian integral over commuting variables z and z^*. Then

$$F_N(\lambda, \mu) = \int \prod_{a=1}^{N} \mathrm{d}z_a^* \, \mathrm{d}z_a \, \mathrm{d}\bar{\theta}_a \, \mathrm{d}\theta_a \langle \exp \mathrm{i}N[\bar{\theta}_a(\lambda\delta_{ab} - X_{ab})\theta_b + z_a^*(\mu\delta_{ab} - X_{ab})z_b]\rangle \tag{10}$$

in which μ is complex, with a small positive imaginary part. The normalization for Grassmann integration which is used here is

$$\int \mathrm{d}\theta \, \mathrm{d}\bar{\theta} \, \theta\bar{\theta} = \frac{1}{\pi}. \tag{11}$$

We also adopt the convention

$$\overline{(\alpha\beta)} = \bar{\alpha}\bar{\beta} \tag{12}$$

which implies

$$\bar{\bar{\alpha}} = -\alpha \tag{13}$$

in order to maintain $\bar{\theta}\theta$ invariant under the bar operation.

Then the Gaussian integration over X is easily performed since

$$\int \mathrm{d}X \, \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\, X^2 + \mathrm{i}N \mathrm{Tr}\, XY} = \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\, Y^2} \tag{14}$$

with $Y_{ji} = -\bar{\theta}_i\theta_j - z_i^* z_j$. This gives

$$\begin{aligned} \mathrm{Tr}\, Y^2 &= \mathrm{Tr}\, Y_{ij} Y_{ji} \\ &= -(\bar{\theta}_i\theta_i)^2 + (z_i^* z_i)^2 + 2(\bar{\theta}_i z_i)(\theta_j z_j^*). \end{aligned} \tag{15}$$

One then introduces auxiliary commuting as well as Grassmann variables

$$\begin{aligned} &\sqrt{\frac{N}{2\pi}} \int \mathrm{e}^{-N\frac{b'^2}{2} - Nb'\bar{\theta}_a\theta_a} \, \mathrm{d}b' = \mathrm{e}^{\frac{N}{2}(\sum_a \bar{\theta}_a\theta_a)^2} \\ &\sqrt{\frac{N}{2\pi}} \int \mathrm{e}^{-N\frac{b^2}{2} - \mathrm{i}Nb z_a^* z_a} \, \mathrm{d}b = \mathrm{e}^{-\frac{N}{2}(\sum_a z_a^* z_a)^2} \\ &\frac{\pi}{N} \int \mathrm{e}^{-N\bar{\eta}\eta + N\eta(\bar{\theta}z) + N\bar{\eta}(z^*\theta)} \, \mathrm{d}\eta \, \mathrm{d}\bar{\eta} = \mathrm{e}^{-N(\bar{\theta}z)(z^*\theta)}. \end{aligned} \tag{16}$$

This allows one to express F_N as

$$\begin{aligned} F_N(\lambda, \mu) = \frac{1}{2} \int \mathrm{d}b \, \mathrm{d}b' \, \mathrm{d}\eta \, \mathrm{d}\bar{\eta} \, \mathrm{e}^{-\frac{N}{2}(b^2 + b'^2 + 2\bar{\eta}\eta)} \\ \times \left[\int \mathrm{d}z \, \mathrm{d}z^* \, \mathrm{d}\theta \, \mathrm{d}\bar{\theta} \, \mathrm{e}^{N(\mathrm{i}\lambda - b')\bar{\theta}\theta + \mathrm{i}N(\mu - b)z^* z} \, \mathrm{e}^{N\bar{\eta}\theta z^* + N\eta\bar{\theta}z} \right]^N. \end{aligned} \tag{17}$$

Two strategies are possible at that stage: either keeping both the commuting and Grassmannian degrees of freedom, and using the supersymmetry of the integral, or integrating out explicitly the Grassmannian variables. They are equally simple for the single ratio of characteristic polynomials considered here; however, for higher correlation functions the use of the invariance under supergroup transformations turns out to be more powerful.

(i) *Integration over the Grassmann variables.* Let us first give the expressions that one finds if one first performs the integrations over $\bar{\theta}, \theta, z^*, z$. This yields

$$\begin{aligned} F_N(\lambda, \mu) &= \frac{1}{2} \int \mathrm{d}b \, \mathrm{d}b' \, \mathrm{d}\eta \, \mathrm{d}\bar{\eta} \, \mathrm{e}^{-\frac{N}{2}(b^2 + b'^2 + 2\bar{\eta}\eta)} \left[\frac{\lambda + \mathrm{i}b'}{\mu - b} + \frac{\bar{\eta}\eta}{(\mu - b)^2} \right]^N \\ &= \left(\frac{N}{2\pi} \right) \int \left[\frac{(\lambda + \mathrm{i}b')^N}{(\mu - b)^N} - \frac{(\lambda + \mathrm{i}b')^{N-1}}{(\mu - b)^{N+1}} \right] \mathrm{e}^{-\frac{N}{2}(b^2 + b'^2)} \, \mathrm{d}b \, \mathrm{d}b'. \end{aligned} \tag{18}$$

It is convenient to use the (appropriately normalized) Hermite polynomials

$$H_n(\lambda) = e^{\lambda^2}\left(-\frac{d}{d\lambda}\right)^n e^{-\lambda^2} = \frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\lambda + it)^n\, e^{-t^2}\, dt \tag{19}$$

which satisfy the recursion formula

$$H_n(\lambda) - 2\lambda H_{n-1}(\lambda) + 2(n-1)H_{n-2}(\lambda) = 0. \tag{20}$$

Then one obtains from (18)

$$F_N(\lambda,\mu) = \frac{1}{2^N\sqrt{\pi}}\left[H_N(\bar{\lambda})\int_{-\infty}^{+\infty} db\frac{e^{-b^2}}{(\bar{\mu}-b)^N} - H_{N-1}(\bar{\lambda})\int_{-\infty}^{+\infty} dt\frac{e^{-b^2}}{(\bar{\mu}-b)^{N+1}}\right] \tag{21}$$

in which μ has a small positive imaginary part and we have used for convenience

$$\bar{\lambda} = \lambda\sqrt{\frac{N}{2}} \qquad \bar{\mu} = \mu\sqrt{\frac{N}{2}}. \tag{22}$$

After repeated integration by parts this gives

$$F_N(\lambda,\mu) = \frac{1}{2^N\sqrt{\pi}(N-1)!}\int_{-\infty}^{+\infty} db\frac{e^{-b^2}}{(\bar{\mu}-b)}[H_N(\bar{\lambda})H_{N-1}(b) - H_{N-1}(\bar{\lambda})H_N(b)] \tag{23}$$

from which follows for an infinitesimal positive imaginary part of μ

$$\mathrm{Im}\,F_N(\lambda,\mu) = -\frac{\sqrt{\pi}}{2^N(N-1)!}\, e^{-\bar{\mu}^2}[H_N(\bar{\lambda})H_{N-1}(\bar{\mu}) - H_{N-1}(\bar{\lambda})H_N(\bar{\mu})] \tag{24}$$

which vanishes as expected for $\lambda = \mu$ since $F_N(\mu,\mu) = 1$.

The density of states $\rho(\lambda)$ is thus given by

$$\rho(\lambda) = -\lim_{\mu\to\lambda}\frac{1}{\pi N}\frac{\partial}{\partial\mu}\mathrm{Im}\,F_N(\lambda,\mu) \tag{25}$$

and one recovers the well-known finite-N expressions [3].

(ii) *Use of supergroup symmetry*. Let us return to (17) and use the supergroup structure. One first obtains

$$-b'\bar{\theta}\theta - ibz^*z + \bar{\eta}\theta z^* + \eta\bar{\theta}z = -i(z^*,\bar{\theta})Q\begin{pmatrix} z \\ \theta \end{pmatrix} \tag{26}$$

with

$$Q = \begin{pmatrix} b & i\bar{\eta} \\ -i\eta & -ib' \end{pmatrix}. \tag{27}$$

Then

$$\tfrac{1}{2}\mathrm{Str}\,Q^2 = b^2 + b'^2 + 2\bar{\eta}\eta \tag{28}$$

in which we have used the supertrace notation 'Str' defined as follows. If one decomposes the supermatrix Q into four blocks

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{29}$$

in which A and D have commuting matrix elements, whereas the matrix elements of C and B are anticommuting. Then by definition

$$\mathrm{Str}\,Q = \mathrm{tr}\,A - \mathrm{tr}\,D. \tag{30}$$

Similarly one defines the superdeterminant

$$\mathrm{S}\det[Q] = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \tag{31}$$

The notation is justified by the fact that

$$\int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}z^*\,\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}\bar{\phi}Q\phi} = (\mathrm{S}\det[Q])^{-1}$$

if $\phi = \binom{z}{\theta}$ and $\bar{\phi} = (z^*, \bar{\theta})$. If one defines the diagonal supermatrix

$$\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}. \tag{32}$$

This leads to the compact expression for $F_N(\lambda, \mu)$ as an integral over the supermatrix Q, which was derived earlier by Zirnbauer [13]

$$\begin{aligned} F_N(\lambda, \mu) &= \frac{1}{2} \int \mathrm{d}Q\,\mathrm{e}^{-\frac{N}{2}\mathrm{Str}\,Q^2} \frac{1}{(\mathrm{S}\det(Q - \Lambda))^N} \\ &= \frac{1}{2}\,\mathrm{e}^{-N(\mu^2-\lambda^2)} \int \mathrm{d}Q\,\mathrm{e}^{-\frac{N}{2}\mathrm{Str}\,Q^2 - N\,\mathrm{Str}\,Q\Lambda} \frac{1}{(\mathrm{S}\det Q)^N}. \end{aligned} \tag{33}$$

One now performs a superunitary transformation to diagonalize Q

$$\begin{pmatrix} z \\ \theta \end{pmatrix} = g \begin{pmatrix} z' \\ \theta' \end{pmatrix} \tag{34}$$

with

$$g = \begin{pmatrix} a & \bar{\beta} \\ \gamma & d \end{pmatrix} \tag{35}$$

in which β and γ are anticommuting. Then with the sign convention used here, $(z^*, \bar{\theta})$ transform by

$$g^\dagger = \begin{pmatrix} a^* & \bar{\gamma} \\ \beta & d^* \end{pmatrix}. \tag{36}$$

Unitary transformations $g^\dagger g = 1$ leave invariant the quadratic form $z^*z + \bar{\theta}\theta$. One may diagonalize the matrix Q by such a unitary transformation

$$Q = g \begin{pmatrix} u & 0 \\ 0 & \mathrm{i}t \end{pmatrix} g^\dagger. \tag{37}$$

Then the integration measure on the matrix Q may be replaced by an integration over the eigenvalues u_1, t and a group integration, up to a Jacobian which is proportional to $1/(u - \mathrm{i}t)^2$. Representation (33) requires then to consider the supergroup equivalent of the HIZ integral defined by

$$I = \int \mathrm{d}g\,\mathrm{e}^{N\,\mathrm{Str}\,gQg^{-1}\Lambda} \tag{38}$$

where Λ and Q are diagonal matrices.

This integral yields for $\lambda \neq \mu$,

$$I = \frac{N}{2\pi^2}(u - \mathrm{i}t)(\mu - \lambda)\,\mathrm{e}^{N(u\mu - \mathrm{i}t\lambda)}. \tag{39}$$

The derivation of this formula relies on a formalism of differential equations for I which will be presented in detail in the coming sections. Note that when $\lambda = \mu$, we have a singular situation. In this case, we have to consider the boundary term [11, 13], and $F_N(\lambda, \mu)$ becomes one as easily seen by the definition. Therefore, we have obtained the representation

$$F_N(\lambda, \mu) = \delta_{\lambda,\mu} + N(\lambda - \mu)\frac{e^{-N(\mu^2-\lambda^2)}}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, du \left(\frac{it}{u}\right)^N \frac{1}{u - it}\, e^{-\frac{N}{2}(u^2+t^2) - iNt\lambda + Nu\mu} \tag{40}$$

where $\delta_{\lambda,\mu} = 1$ for $\lambda = \mu$, and zero otherwise.

The density of states is then recovered through

$$\begin{aligned}\rho(\lambda) &= -\mathrm{Im} \lim_{\mu\to\lambda} \frac{\partial}{\partial\mu}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\oint\frac{du}{2\pi}\, du\left(\frac{-it}{u}\right)^N\frac{1}{u - it}\, e^{-\frac{N}{2}(u^2+t^2) - iNt\lambda + Nu\mu} \\ &= \frac{1}{N}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\oint\frac{du}{2i\pi}\left(\frac{-it}{u}\right)^N\frac{1}{u - it}\, e^{-\frac{N}{2}(u^2+t^2) - iNt\lambda + Nu\lambda}\end{aligned} \tag{41}$$

where the contour integral of u circles around the origin. This exact representation, valid for any N, has been derived earlier by a completely different method [16], without supersymmetry. The semi-circle law is recovered from there easily, in the large-N limit, by a saddle-point integration (it is simpler for that to consider $\partial\rho/\partial\lambda$).

3. Single ratio of characteristic polynomials in the GOE ensemble

We now consider a single ratio of characteristic polynomials of GOE, defined as

$$F_N(\lambda, \mu) = \left\langle \frac{\det(\lambda - X)}{\det(\mu - X)} \right\rangle. \tag{42}$$

The averages $\langle \cdots \rangle$ are computed with respect to the Gaussian distribution

$$P(X) = \frac{1}{Z}\exp\left(-\frac{N}{2}\mathrm{Tr}\, X^2\right) \tag{43}$$

where the matrix X is an $N \times N$ real symmetric matrix.

As for the GUE, the ratio in (42) is expressed by

$$F_N(\lambda, \mu) = \int \prod_{a=1}^{N} dz_a^*\, dz_a\, d\bar{\theta}_a\, d\theta_a \langle \exp iN[\bar{\theta}_a(\lambda\delta_{ab} - X_{ab})\theta_b + z_a^*(\mu\delta_{ab} - X_{ab})z_b]\rangle \tag{44}$$

in which μ is complex, with a small positive imaginary part. We use the normalization for Grassmannian integration of the previous section.

Then the Gaussian integration over X is easily performed since

$$\int dX\, e^{-\frac{N}{2}\mathrm{Tr}\, X^2 + iN\,\mathrm{Tr}\, XY} = e^{-\frac{N}{4}\mathrm{Tr}(Y^2 + YY^T)} \tag{45}$$

with $Y_{ji} = -\bar{\theta}_i\theta_j - z_i^* z_j$. This gives

$$\begin{aligned}\mathrm{Tr}\, Y^2 &= \mathrm{Tr}\, Y_{ij}Y_{ji} \\ &= -(\bar{\theta}_i\theta_i)^2 + (z_i^* z_i)^2 + 2(\bar{\theta}_i z_i)(\theta_j z_j^*)\end{aligned} \tag{46}$$

$$\mathrm{Tr}\, YY^T = 2(\bar{\theta}_i z_i^*)(z_j\theta_j) + (z_i^* z_i^*)(z_j z_j). \tag{47}$$

One then introduces auxiliary commuting as well as Grassmann variables

$$\sqrt{\frac{N}{\pi}}\int \mathrm{e}^{-Nb'^2-Nb'\bar\theta_a\theta_a}\,\mathrm{d}b' = \mathrm{e}^{\frac{N}{4}(\sum_a\bar\theta_a\theta_a)^2} \qquad \sqrt{\frac{N}{\pi}}\int \mathrm{e}^{-Nb^2-\mathrm{i}Nbz_a^*z_a}\,\mathrm{d}b = \mathrm{e}^{-\frac{N}{4}(\sum_a z_a^*z_a)^2}$$
$$\frac{N}{\pi}\int \mathrm{e}^{-Nu^*u+\frac{\mathrm{i}N}{2}uz_a^*z_a^*+\frac{\mathrm{i}N}{2}u^*z_az_a}\,\mathrm{d}u^*\,\mathrm{d}u = \mathrm{e}^{-\frac{N}{4}(z_a^*z_a^*)(z_bz_b)}$$
$$\frac{\pi}{2N}\int \mathrm{e}^{-2N\bar\eta\eta+N\eta(\bar\theta z)+N\bar\eta(z^*\theta)}\,\mathrm{d}\bar\eta\,\mathrm{d}\eta = \mathrm{e}^{-\frac{N}{2}(\bar\theta z)(z^*\theta)} \tag{48}$$
$$\frac{\pi}{2N}\int \mathrm{e}^{-2N\bar\eta'\eta'+N\eta'(\bar\theta z^*)+N\bar\eta'(z\theta)}\,\mathrm{d}\bar\eta'\,\mathrm{d}\eta' = \mathrm{e}^{-\frac{N}{2}(\bar\theta z^*)(z\theta)}.$$

This allows F_N to be expressed as

$$F_N(\lambda,\mu) = \frac{1}{4}\int \mathrm{d}b\,\mathrm{d}b'\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}\bar\eta\,\mathrm{d}\eta\,\mathrm{d}\bar\eta'\,\mathrm{d}\eta'\,\mathrm{e}^{-N(b^2+b'^2+u^*u+2\bar\eta\eta+2\bar\eta'\eta')}$$
$$\times\left[\int \mathrm{d}z\,\mathrm{d}z^*\,\mathrm{d}\bar\theta\,\mathrm{d}\theta\,\mathrm{e}^{N(\mathrm{i}\lambda-b')\bar\theta\theta+\mathrm{i}N(\mu-b)z^*z}\,\mathrm{e}^{N\bar\eta\theta z^*+N\eta\bar\theta z+\frac{\mathrm{i}}{2}N(u^*z^2+uz^{*2})+N\eta' z^*\bar\theta+N\bar\eta'\theta z}\right]^N. \tag{49}$$

Two strategies are possible, as explained in the previous section. Let us first give the expressions that one finds if one first performs the integrations over $\bar\theta, \theta, z^*, z$. Noting that

$$\int \mathrm{e}^{\mathrm{i}N(\mu-b)z^*z+\frac{\mathrm{i}}{2}N(u^*z^2+uz^{*2})}\,\mathrm{d}z\,\mathrm{d}z^* = \frac{\mathrm{i}\pi}{N\sqrt{(\mu-b)^2-|u|^2}} \tag{50}$$

the integration over $z, z^*, \bar\theta, \theta$ yields

$$F_N(\lambda,\mu) = \frac{1}{4}\int \mathrm{d}b\,\mathrm{d}b'\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}\bar\eta\,\mathrm{d}\eta\,\mathrm{d}\bar\eta'\,\mathrm{d}\eta'\,\mathrm{e}^{-N(b^2+b'^2+u^*u+2\bar\eta\eta+2\bar\eta'\eta')}\left[\frac{\lambda+\mathrm{i}b'}{((\mu-b)^2-|u|^2)^{1/2}}\right.$$
$$\left. - \frac{1}{((\mu-b)^2-|u|^2)^{3/2}}\{u\bar\eta'\eta+u^*\bar\eta\eta'-(\mu-b)(\bar\eta\eta+\bar\eta'\eta')\}\right]^N. \tag{51}$$

One can next integrate over the remaining Grassmann variables $\bar\eta$, η, $\bar\eta'$, η', and denoting $D=\sqrt{(\mu-b)^2-|u|^2}$ one obtains at the end

$$F_N(\lambda,\mu) = \left(\frac{N}{\pi}\right)^2\int\frac{1}{D^N}\left[(\lambda+\mathrm{i}b')^N - \frac{1}{D^2}(\mu-b)(\lambda+\mathrm{i}b')^{N-1}+\frac{N-1}{4ND^2}(\lambda+\mathrm{i}b')^{N-2}\right]$$
$$\times\,\mathrm{e}^{-N(b^2+b'^2+u^*u)}\,\mathrm{d}b\,\mathrm{d}b'\,\mathrm{d}u\,\mathrm{d}u^*. \tag{52}$$

It is convenient to use the (appropriately normalized) Hermite polynomials defined by (19). Then, using a few integrations by parts, one obtains from (52)

$$F_N(\lambda,\mu) = -\frac{(N-1)}{2^{N-1}\pi}H_{N-2}(\bar\lambda)\int_{-\infty}^{+\infty}\frac{\mathrm{d}b}{\sqrt{\pi}}\frac{\mathrm{e}^{-b^2}}{(\bar\mu-b)^N}+\frac{1}{2^{N-1}\pi}H_{N-1}(\bar\lambda)$$
$$\times\int_{-\infty}^{+\infty}\frac{\mathrm{d}b}{\sqrt{\pi}}\,\mathrm{e}^{-b^2}(\bar\lambda-2b)\int_0^{\infty}\mathrm{d}\rho\,\mathrm{e}^{-\rho}\frac{1}{[(\bar\mu-b)^2-\rho]^{N/2}} \tag{53}$$

in which μ has a small positive imaginary part and we have used for convenience

$$\bar\lambda = \lambda\sqrt{N} \qquad \bar\mu = \mu\sqrt{N}. \tag{54}$$

Relation (53) for the average ratio allows one to find in particular the density of states, and the density of states $\rho(\lambda)$ is thus

$$\rho(\lambda) = -\lim_{\mu\to\lambda}\frac{1}{\pi N}\frac{\partial}{\partial\mu}\mathrm{Im}\,F_N(\lambda,\mu) \tag{55}$$

when $\mathrm{Im}\,\lambda\to 0$.

The explicit formula (53) allows one to recover standard results derived by the method of skew-orthogonal polynomials [3]. In order to write explicitly the imaginary part of F_N one has to distinguish between even and odd N. Let us, for instance, work with $N = 2M$ and check (53) for $N = 2$ and $N = 4$. Using

$$\mathrm{Im} \int_{-\infty}^{+\infty} \mathrm{d}b \frac{\mathrm{e}^{-b^2}}{(\bar{\mu} - b)^N} = -\frac{\pi}{(N-1)!} H_{N-1}(\bar{\mu})\,\mathrm{e}^{-\bar{\mu}^2} \tag{56}$$

and

$$\mathrm{Im} \int_{0}^{+\infty} \mathrm{d}\rho \frac{\mathrm{e}^{-\rho}}{(\bar{\mu} - b)^2 - \rho} = -\pi\,\mathrm{sgn}(\bar{\mu} - b)\,\mathrm{e}^{-(\bar{\mu}-b)^2} \tag{57}$$

where $\mathrm{sgn}(x) = +1$ for $x > 0$ and -1 for $x < 0$, one thus finds for $N = 2$

$$\sqrt{\pi}\,\mathrm{Im}\,F_2(\lambda, \mu) = (\bar{\mu} - \bar{\lambda})\,\mathrm{e}^{-\bar{\mu}^2} - \bar{\lambda}(\bar{\mu} - \bar{\lambda})B(\bar{\mu}) \tag{58}$$

in which

$$B(x) = \mathrm{e}^{-x^2/2} \int_0^x \mathrm{d}y\,\mathrm{e}^{-y^2/2}. \tag{59}$$

For $N = 4$ we use

$$\int_0^\infty \frac{1}{((\mu - b)^2 - \rho)^2}\,\mathrm{e}^{-\rho}\,\mathrm{d}\rho = \frac{1}{(\mu - b)^2} + \int_0^\infty \frac{1}{(\mu - b)^2 - \rho}\,\mathrm{e}^{-\rho}\,\mathrm{d}\rho \tag{60}$$

and obtain

$$\begin{aligned}\sqrt{\pi}\,\mathrm{Im}\,F_4(\lambda, \mu) &= \tfrac{1}{16}\,\mathrm{e}^{-\bar{\mu}^2}[H_2(\bar{\lambda})H_3(\bar{\mu}) - H_3(\bar{\lambda})H_2(\bar{\mu})] \\ &\quad + \tfrac{1}{8}(\bar{\mu} - \bar{\lambda})H_3(\bar{\lambda})(2\bar{\mu}\,\mathrm{e}^{-\bar{\mu}^2} + B(\bar{\mu})).\end{aligned} \tag{61}$$

We see that, as it should, $\mathrm{Im}\,F_N(\mu, \mu)$ vanishes (since $F_N(\mu, \mu) = 1$). The above expressions agree with those derived from orthogonal polynomials,

$$\begin{aligned}S_2(\lambda, \mu) &= \phi_0(\lambda)\phi_0(\mu) - \phi_0{}'(\lambda) \int_0^\mu \phi_0(t)\,\mathrm{d}t \\ &= \phi_0(\lambda)\phi_0(\mu) + \phi_1(\lambda)\phi_1(\mu) + \phi_1(\lambda) \int_0^\mu \phi_2(t)\,\mathrm{d}t\end{aligned} \tag{62}$$

where $\phi_n(\lambda) = (2^n n! \sqrt{\pi})^{-1/2}\,\mathrm{e}^{-\lambda^2/2} H_n(\lambda)$. We find $\mathrm{Im}\,F_2 = (\mu - \lambda)S_2(\lambda, \mu)\,\mathrm{e}^{-\mu^2/2+\lambda^2/2}$. The $S_2(\lambda, \mu)$ is the diagonal part of a kernel, which is a 2×2 quaternion matrix [3].

For general N, $S_N(\lambda, \mu)$ is given by

$$\begin{aligned}S_N(\lambda, \mu) &= \sum_{i=0}^{N-1} \phi_i(\lambda)\phi_i(\mu) + \left(\frac{N}{2}\right)^{1/2} \phi_{N-1}(\lambda) \int_0^\mu \phi_N(t)\,\mathrm{d}t \\ &= \sum_{i=0}^{\frac{N}{2}-1} \phi_{2i}(\lambda)\phi_{2i}(\mu) - \sum_{i=0}^{\frac{N}{2}-1} \phi'_{2i}(\lambda) \int_0^\mu \phi_{2i}(t)\,\mathrm{d}t.\end{aligned} \tag{63}$$

In the large-N limit, the second term in (63) in the first line is negligible compared to the first one. Indeed, since $\phi_N(t)$ is a rapidly oscillating function, its integral in the second term is asymptotically small. Then one recovers the semi-circle law in the limit $\lambda \to \mu$. Therefore the first term of $S_N(\lambda, \mu)$ is dominating; it coincides with the well-known universal kernel for the GUE ensemble.

Let us further check for the $N = 4$ case. In this case, we obtain

$$\begin{aligned}F_4(\lambda, \mu) &= \tfrac{1}{16}[H_4(\lambda) - 2\mu H_3(\lambda) + 6H_2(\lambda)]B \\ &\quad + \left[\tfrac{1}{8}\mu H_4(\lambda) + \tfrac{1}{8}(4\mu^2 - 1)H_3(\lambda) - \tfrac{1}{2}\mu^3 H_2(\lambda)\right]\mathrm{e}^{-\lambda^2}.\end{aligned} \tag{64}$$

Then we have

$$\lim_{\lambda\to\mu}\frac{\sqrt{\pi}}{\lambda-\mu}\,\mathrm{Im}\,F_4(\lambda,\mu) = -\left(\bar{\mu}^3-\frac{3}{2}\bar{\mu}\right)B(\bar{\mu}) - \left(3\bar{\mu}^2+\frac{3}{2}\right)\mathrm{e}^{-\mu^2}. \qquad (65)$$

The right-hand side of this equation agrees precisely with $S_4(\mu,\mu)$ in (63). This allows one to check also the density of states since

$$\mathrm{Im}\lim_{\lambda\to\mu}\frac{\partial}{\partial\mu}F_N(\lambda,\mu) = -S(\mu,\mu). \qquad (66)$$

We have verified the known result of the density of states in the GOE, $\rho(\lambda)=\frac{1}{\pi N}S(\lambda,\lambda)$.

We are interested in the large-N limit and the choice of variables b, b' and ρ above is not the best one. Let us define instead u_1, u_2 and t (analogous to the variables u and t which have been used for the unitary case [16, 17]; a comparison with the unitary case is very useful) as follows

$$b=\frac{u_1+u_2}{2} \qquad \rho=\frac{(u_1-u_2)^2}{4} \qquad (67)$$

and $b'=t$. The geometric meaning of those variables will become clear later when the formalism of graded supermatrices is introduced. Then we have from (52)

$$F_N(\lambda,\mu)=\frac{N^2}{2\pi^2}\iiint_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{e}^{-N(t+\mathrm{i}\lambda)^2-\frac{N}{2}(u_1+\mu)^2-\frac{N}{2}(u_2+\mu)^2}\frac{(\mathrm{i}t)^N}{(u_1u_2)^{N/2}}|u_1-u_2|$$
$$\times\left[1+\frac{1}{2}\frac{u_1+u_2}{\mathrm{i}t(u_1u_2)}+\frac{(N-1)}{4N}\frac{1}{(\mathrm{i}t)^2u_1u_2}\right]. \qquad (68)$$

This representation is well adapted to the large-N limit. The saddle points are

$$t^c=\frac{-\mathrm{i}\lambda\pm\sqrt{2-\lambda^2}}{2}=t_+,t_-$$
$${u_1}^c=\frac{\mu\pm\mathrm{i}\sqrt{2-\mu^2}}{2}=u_+,u_- \qquad (69)$$
$${u_2}^c=\frac{\mu\pm\mathrm{i}\sqrt{2-\mu^2}}{2}.$$

We have previously analysed a similar large-N behaviour for the kernel of the unitary case [18]. The factor $(\mathrm{i}t/\sqrt{u_1u_2})^N$ is oscillatory in the large-N limit unless we take the pair of saddle points with $(t,u_1,u_2)=(t_+,u_+,u_+)$ and (t_-,u_-,u_-). The other six possibilities are sub-leading. The fluctuations around the saddle point give

$$\frac{1}{(f''(t_c))^{1/2}}=\frac{1}{\sqrt{1-\frac{1}{2t^2}}} \qquad (70)$$

and for u_1 and u_2 we have similar Gaussian fluctuations to take into account, together with the factor $|u_1-u_2|$. Then the Gaussian integration for the u-fluctuations yields

$$\frac{1}{(f''(u_c))^{3/2}}=\frac{1}{\left(1-\frac{1}{2u^2}\right)^{3/2}}. \qquad (71)$$

The second term in the integrand of (68) becomes at the saddle point

$$1+\frac{1}{2}\frac{u_1+u_2}{\mathrm{i}t(u_1u_2)}+\frac{1}{4}\frac{1}{(\mathrm{i}t)^2u_1u_2}=1+\frac{1}{\mathrm{i}t_cu_c}+\frac{1}{4(\mathrm{i}t_c)^2u_c^2} \qquad (72)$$

which is $\left(1+\frac{1}{2\mathrm{i}t_c u_c}\right)^2$. Noting that $u_c = \mathrm{i}t_c$, we find that this factor cancels the results (70) and (71) of the Gaussian fluctuations. In the large-N limit, then we have

$$\mathrm{Im}\, F_N(\lambda,\mu) = \sin\left(N(\lambda-\mu)\sqrt{2-\lambda^2}\right) \tag{73}$$

$$\rho(\lambda) = \frac{1}{\pi N}\lim_{\mu\to\lambda}\frac{\partial}{\partial\lambda}\sin\left[N(\lambda-\mu)\sqrt{2-\lambda^2}\right]$$
$$= \frac{1}{\pi}\sqrt{2-\lambda^2} \tag{74}$$

as expected.

4. Supermatrix diagonalization

We have obtained an integral representation of the averaged ratio of characteristic polynomials (51) by a Gaussian integration over θ and z. If, instead of integrating out the anticommuting variables, we keep bosonic and fermionic (supersymmetric) degrees of freedom, the formulation becomes more transparent. Furthermore, it turns out that the diagonalization of the supermatrices by a supergroup element provides an extremely useful representation in terms of the eigenvalues.

Let us define the supermatrix Q by

$$Q = \begin{pmatrix} b & -u & \mathrm{i}\bar\eta & \mathrm{i}\eta' \\ -u^* & b & \mathrm{i}\bar\eta' & \mathrm{i}\eta \\ -\mathrm{i}\eta & -\mathrm{i}\eta' & -\mathrm{i}b' & 0 \\ \mathrm{i}\bar\eta' & \mathrm{i}\bar\eta & 0 & -\mathrm{i}b' \end{pmatrix} \tag{75}$$

which is such that one can write the various quantities which appear in the integral representation (49) as follows:

$$-\mathrm{i}bz^*z + \frac{\mathrm{i}}{2}(u^*z^2 + uz^{*2}) + \bar\eta\theta z^* + \eta\bar\theta z + \eta' z^*\bar\theta + \bar\eta'\theta z - b'\bar\theta\theta$$
$$= -\frac{\mathrm{i}}{2}(z^*, z, \bar\theta, -\theta)\, Q \begin{pmatrix} z \\ z^* \\ \theta \\ \bar\theta \end{pmatrix} \tag{76}$$

and

$$\tfrac{1}{2}\mathrm{Str}\, Q^2 = b^2 + b'^2 + u^*u + 2\bar\eta\eta + 2\bar\eta'\eta'. \tag{77}$$

From the definition (31) of Sdet[Q], one has

$$\mathrm{Sdet}[Q] = -\frac{b^2 - u^*u}{\left[b' + \frac{\mathrm{i}}{b^2-u^*u}\left(b(\bar\eta\eta + \bar\eta'\eta') + u\bar\eta'\eta + u^*\bar\eta\eta'\right)\right]^2}. \tag{78}$$

Similarly we define the diagonal supermatrix Λ, with elements $(\mu, \mu, \lambda, \lambda)$ on the diagonal. This leads to a compact expression for $F_N(\lambda,\mu)$ as an integral over the supermatrix Q,

$$F_N(\lambda,\mu) = \frac{1}{4}\,\mathrm{e}^{N(\mu^2-\lambda^2)}\int \mathrm{d}Q\,\mathrm{e}^{-\frac{N}{2}\,\mathrm{Str}\,Q^2 - \mathrm{i}N\,\mathrm{Str}\,Q\Lambda}\frac{1}{(\mathrm{Sdet}Q)^{N/2}} \tag{79}$$

which is identical to (33).

Superunitary transformations on the variables are defined by

$$\begin{pmatrix} z \\ z^* \\ \theta \\ \bar\theta \end{pmatrix} = g \begin{pmatrix} z' \\ z'^* \\ \theta' \\ \bar\theta' \end{pmatrix} \tag{80}$$

with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{81}$$

in which b and c are 2×2 matrices with anticommuting elements. Then

$$g^\dagger = \begin{pmatrix} a^\dagger & \bar c \\ -\bar b & d^\dagger \end{pmatrix}. \tag{82}$$

Unitary transformations $g^\dagger g = 1$ leave invariant the quadratic form $z^* z + \bar\theta\theta$. One may diagonalize the matrix Q by such a unitary transformation

$$Q = g \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \mathrm{i}t & \\ & & & \mathrm{i}t \end{pmatrix} g^\dagger \tag{83}$$

since one can see, through an identity similar to (78), that it has one doubly degenerate eigenvalue. Then the integration measure on the matrix Q may be replaced by an integration over the eigenvalues u_1, u_2, t and a group integration, up to a Jacobian which may be found as follows. One considers a group element near the identity

$$g = 1 + \mathrm{i}\epsilon \tag{84}$$

with

$$\epsilon = \begin{pmatrix} \mathrm{i}a & b \\ \bar b & 0 \end{pmatrix} \tag{85}$$

with

$$a = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}. \tag{86}$$

Starting from a diagonal matrix Q it is transformed under this operation into $Q + \delta Q$, with

$$\delta Q = \mathrm{i}[\epsilon, Q] = \mathrm{i} \begin{pmatrix} 0 & v(u_2 - u_1) & (\mathrm{i}t - u_1)b_{11} & (\mathrm{i}t - u_1)b_{12} \\ v^*(u_1 - u_2) & 0 & (\mathrm{i}t - u_2)b_{21} & (\mathrm{i}t - u_2)b_{22} \\ -(\mathrm{i}t - u_1)\bar b_{11} & -(\mathrm{i}t - u_2)\bar b_{21} & 0 & 0 \\ -(\mathrm{i}t - u_1)\bar b_{12} & -(\mathrm{i}t - u_2)\bar b_{22} & 0 & 0 \end{pmatrix}. \tag{87}$$

The Jacobian J is then simply

$$J = \frac{|u_1 - u_2|}{(\mathrm{i}t - u_1)^2(\mathrm{i}t - u_2)^2}. \tag{88}$$

The representation (79) leads to consideration of the supergroup equivalent of the HIZ integral defined by

$$I = \int \mathrm{d}g\, \mathrm{e}^{N \,\mathrm{Str}\, g Q g^{-1} \Lambda} \tag{89}$$

where Λ and Q are diagonal matrices, $\Lambda = \mathrm{diag}(\mu, \mu, \lambda, \lambda)$ and $Q = \mathrm{diag}(u_1, u_2, \mathrm{i}t, \mathrm{i}t)$. (We are still dealing here with a single ratio (42) of characteristic polynomials and shall return to this point for higher correlation functions.) The HIZ integral I may be obtained as the solution of a heat kernel differential equation. Indeed, it satisfies the equation

$$\Delta_Q I(Q) = \epsilon I(Q) \tag{90}$$

in which Δ is the Laplacian with respect to the matrix elements of Q, sum of second derivatives with respect to the five commuting variables, and to the four Grassmannian ones, and

$$\epsilon = 2N^2(\mu^2 - \lambda^2). \tag{91}$$

Since the integral I is invariant under supergroup transformations of Q, it is a function of the three distinct eigenvalues of Q, u_1, u_2, $\mathrm{i}t$, and one may express the Laplacian as a second-order differential operator with respect to those eigenvalues:

$$\frac{1}{2J}\frac{\partial}{\partial t}J\frac{\partial}{\partial t}I(Q) + \frac{1}{J}\sum_{\alpha=1}^{2}\frac{\partial}{\partial u_\alpha}J\frac{\partial}{\partial u_\alpha}I(Q) = -\epsilon I(Q). \tag{92}$$

(The factor $1/2$ comes from the degeneracy of the eigenvalue $\mathrm{i}t$.) Let h denote the square root of the Jacobian

$$h = J^{1/2} \tag{93}$$

and substitute in the differential equation $I(Q) = \chi/h$. Then, with the help of the identity

$$\frac{1}{2h}\frac{\partial}{\partial t}\left(h^2\frac{\partial}{\partial t}\frac{\chi}{h}\right) + \frac{1}{h}\sum_{\alpha=1}^{2}\frac{\partial}{\partial u_\alpha}\left(h^2\frac{\partial}{\partial u_\alpha}\frac{\chi}{h}\right) = \frac{1}{2}\frac{\partial^2\chi}{\partial t^2} + \sum_{\alpha=1}^{2}\frac{\partial^2\chi}{\partial {u_\alpha}^2} - \frac{\chi}{2h}\frac{\partial^2 h}{\partial t^2} - \frac{\chi}{h}\sum_{\alpha=1}^{2}\frac{\partial^2 h}{\partial {u_\alpha}^2} \tag{94}$$

the first derivative terms of χ cancel in the differential equation and one obtains, up to contact terms which have a vanishing contribution to the final result,

$$\frac{1}{2}\frac{\partial^2\chi}{\partial t^2} + \sum_{\alpha=1}^{2}\frac{\partial^2\chi}{\partial {u_\alpha}^2} + \left[-\sum_{\alpha=1}^{2}\frac{1}{(\mathrm{i}t-u_\alpha)^2} + \frac{1}{2}\frac{1}{(u_1-u_2)^2}\right]\chi = -\epsilon\chi. \tag{95}$$

If one substitutes

$$\chi = \mathrm{e}^{-2\mathrm{i}Nt\lambda + Nu_1\mu + Nu_2\mu}\sqrt{|u_1-u_2|}\,g \tag{96}$$

(the square root factor comes from the degeneracies in the matrix Λ), the pole $1/(u_1-u_2)$ in (95) is cancelled. Then the HIZ integral I becomes

$$I = \mathrm{e}^{-2\mathrm{i}N\lambda t + N(u_1+u_2)\mu}(\mathrm{i}t-u_1)(\mathrm{i}t-u_2)(\lambda-\mu)^2 g \tag{97}$$

in which g satisfies the Laplacian (heat kernel) equation

$$\begin{aligned}
-2\mathrm{i}N\lambda\frac{\partial g}{\partial t} &+ 2N\mu\frac{\partial g}{\partial u_1} + 2N\mu\frac{\partial g}{\partial u_2} + \frac{1}{2}\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g}{\partial {u_1}^2} + \frac{\partial^2 g}{\partial {u_2}^2} + \frac{1}{u_1-u_2}\left(\frac{\partial g}{\partial u_1} - \frac{\partial g}{\partial u_2}\right) \\
&+ \left[-\frac{1}{(\mathrm{i}t-u_1)^2} - \frac{1}{(\mathrm{i}t-u_2)^2}\right] g = 0.
\end{aligned} \tag{98}$$

Remarkably enough, the solution of this equation is simply

$$g = 1 - \frac{1}{2N(\mathrm{i}t-u_1)(\lambda-\mu)} - \frac{1}{2N(\mathrm{i}t-u_2)(\lambda-\mu)}. \tag{99}$$

Collecting all factors from (97), together with the Jacobian (88), $F_N(\lambda,\mu)$ may then be written as

$$\begin{aligned}
F_N(\lambda,\mu) = \int &\frac{(\mathrm{i}t)^N}{(u_1u_2)^{\frac{N}{2}}}\frac{|u_1-u_2|}{(\mathrm{i}t-u_1)(\mathrm{i}t-u_2)}(\lambda-\mu)^2\left[1 - \frac{1}{2N(\lambda-\mu)}\left(\frac{1}{\mathrm{i}t-u_1} + \frac{1}{\mathrm{i}t-u_2}\right)\right] \\
&\times \mathrm{e}^{-2\mathrm{i}Nt\lambda + N(u_1+u_2)\mu - Nt^2 - \frac{N}{2}(u_1^2+u_2^2)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2 + \delta_{\lambda,\mu}.
\end{aligned} \tag{100}$$

From there one derives the density of states

$$\rho(\lambda) = \frac{1}{\pi N}\mathrm{Im}\lim_{\mu\to\lambda}\frac{\partial}{\partial\lambda}F_N(\lambda,\mu)$$
$$= -\frac{1}{8\pi^2 N}\,\mathrm{Im}\lim_{\mu\to\lambda}\int \mathrm{d}t\,\mathrm{d}u\frac{(\mathrm{i}t)^N}{(u_1u_2)^{\frac{N}{2}}}\frac{|u_1-u_2|}{(\mathrm{i}t-u_1)(\mathrm{i}t-u_2)}\left[\frac{1}{\mathrm{i}t-u_1}+\frac{1}{\mathrm{i}t-u_2}\right]$$
$$\times\,\mathrm{e}^{-2\mathrm{i}Nt\lambda+N(u_1+u_2)\mu-Nt^2-\frac{N}{2}(u_1^2+u_2^2)}. \qquad (101)$$

This result has also been obtained recently by Guhr and Kohler [15]. Since this expression is quite different from (68), derived in section 2, it is worth showing that it does give the same result for the density of states. For simplicity let us consider the $N = 2$ case.

We return to the variables defined in (67); then $\rho(\lambda)$ reads

$$\rho(\lambda) = \frac{1}{8\pi N}\mathrm{Im}\int\frac{(\mathrm{i}t)^2}{b^2-\rho}\frac{(\mathrm{i}t-b)}{((\mathrm{i}t)^2-2\mathrm{i}tb+b^2-\rho)^2}\,\mathrm{e}^{-N(t+\mathrm{i}\lambda)^2-N(b-\lambda)^2-N\rho}\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}\rho. \qquad (102)$$

The imaginary part is evaluated as in (56),

$$\rho = \frac{1}{8\pi N}\int \mathrm{sgn}(b)\delta(\rho-b^2)\frac{(\mathrm{i}t)^2(\mathrm{i}t-b)}{((\mathrm{i}t)^2-2\mathrm{i}tb+b^2-\rho)^2}\,\mathrm{e}^{-N(t+\mathrm{i}\lambda)^2-N(b-\lambda)^2-N\rho}\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}\rho$$
$$= \frac{1}{8\pi N}\int \mathrm{sgn}(b)\frac{\mathrm{i}t-b}{(\mathrm{i}t-2b)^2}\,\mathrm{e}^{-N(t+\mathrm{i}\lambda)^2-Nb^2-N(b-\lambda)^2}\,\mathrm{d}b\,\mathrm{d}t. \qquad (103)$$

The factor $\mathrm{i}t - b$ in the integrand is equivalent to taking a derivative with respect to λ of the exponent. We also introduce an auxiliary integral over a variable s to express the denominator,

$$\rho = \frac{1}{2}\frac{\partial}{\partial\lambda}\left[\int_{-\infty}^{\infty}\mathrm{d}t\int_0^{\infty}\mathrm{d}b\int_0^{\infty}\mathrm{d}s\,s\,\mathrm{e}^{-s(2b-\mathrm{i}t)-Nb^2-N(b-\lambda)^2-N(t+\mathrm{i}\lambda)^2}\right.$$
$$\left.-\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{0}\mathrm{d}b\int_0^{\infty}\mathrm{d}s\,s\,\mathrm{e}^{s(2b-\mathrm{i}t)-Nb^2-N(b-\lambda)^2-N(t+\mathrm{i}\lambda)^2}\right]. \qquad (104)$$

This difference of integrals is an even function of λ, since it may be expressed as

$$\rho = \frac{1}{2}\frac{\partial}{\partial\lambda}[f(\lambda)-f(-\lambda)] \qquad (105)$$

with f, after integration over t, being

$$f(\lambda) = \int_0^{\infty}\int_0^{\infty}\mathrm{d}s\,\mathrm{d}b\,s\,\mathrm{e}^{-2N(b-\frac{\lambda}{2}+\frac{s}{2N})^2+\frac{s^2}{4N}-\frac{N}{2}\lambda^2} = \int_0^{\infty}\mathrm{d}s\,s\,\mathrm{e}^{\frac{s^2}{4N}-\frac{N}{2}\lambda^2}\int_{-\frac{\lambda}{2}+\frac{s}{2N}}^{\infty}\mathrm{d}b\,\mathrm{e}^{-2b^2}. \qquad (106)$$

Therefore, one obtains

$$\frac{\partial}{\partial\lambda}f = \int_0^{\infty}s\,\mathrm{e}^{-\frac{1}{4N}(s-2N\lambda)^2}\,\mathrm{d}s = 2N\,\mathrm{e}^{-N\lambda^2}+N\lambda\sqrt{4N\pi}+2N\lambda\int_{-2N\lambda}^{0}\mathrm{e}^{-\frac{z^2}{4N}}\,\mathrm{d}z. \qquad (107)$$

Dropping the odd terms in λ, we recover the expression (58) for the $N = 2$ density of states. For general N the same method applies. Taking the imaginary part of $\frac{1}{(b^2-\rho)^{(N/2)}}$ gives derivatives of the delta function, and using integration by parts, we obtain the expression for the density of states.

5. Large-N limit: the semi-circle law

We now consider the large-N limit of the density of states, based on the representation (101). In the large-N limit, one may use the saddle-point method for the integration over t and

u_1, u_2. Since the denominators in (101) have double poles, we again change variables to $b = \frac{u_1+u_2}{2}$, $r = \frac{(u_1-u_2)^2}{4}$. Then the integral becomes

$$\rho(\lambda) = \lim_{\mu\to\lambda} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}b \int_0^{\infty} \mathrm{d}r \frac{(\mathrm{i}t)^N}{(b^2-r)^{N/2}} \frac{2\mathrm{i}t - 2b}{[(\mathrm{i}t)^2 - 2b(\mathrm{i}t) + b^2 - r]^2} \times \mathrm{e}^{-N(t+\mathrm{i}\lambda)^2 - Nb^2 + 2Nb\mu - Nr}. \tag{108}$$

Integrating by parts over r, one is led to

$$\rho(\lambda) = -\frac{1}{8\pi^2} \lim_{\mu\to\lambda} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}b \left(\frac{\mathrm{i}t}{b}\right)^N \frac{1}{\mathrm{i}t - b} \mathrm{e}^{-N(t+\mathrm{i}\lambda)^2 - N(b-\mu)^2}$$
$$+ \frac{N}{4\pi^2} \lim_{\mu\to\lambda} \mathrm{Im} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}u_1\, \mathrm{d}u_2 \frac{(\mathrm{i}t)^N}{(u_1u_2)^{N/2}} \frac{|u_1-u_2|}{\mathrm{i}t - u_1}\left(1 - \frac{1}{2u_1u_2}\right) \times \mathrm{e}^{-N(t+\mathrm{i}\lambda)^2 - \frac{N}{2}(u_1-\mu)^2 - \frac{N}{2}(u_2-\mu)^2}. \tag{109}$$

In the second term, we have used again the variables u_1 and u_2 instead of b and ρ. The saddle points are

$$t_\pm = \frac{-\mathrm{i}\lambda \pm \sqrt{2-\lambda^2}}{2} \tag{110}$$

$$b_\pm = u_\pm = \frac{\mu \pm \sqrt{\mu^2 - 2}}{2}. \tag{111}$$

The two leading saddle points are (t_+, b_+) and (t_-, b_-). Other choices such as (t_+, b_-) give an oscillatory behaviour, which can be neglected in the large-N limit. The first term in (109) is the same as GUE. For the second term, one needs to evaluate the Gaussian fluctuation around the saddle point since there is a factor $|u_1 - u_2|$. One retains also the pair of saddle points (t_+, u_{1+}, u_{2+}) and (t_-, u_{1-}, u_{2-}). The fluctuation around the saddle points gives a factor $1/\sqrt{f''(t_c)} = 1\Big/\sqrt{1 + \frac{1}{2t_c^2}} = \frac{\sqrt{t_c}}{(2-\lambda^2)^{1/4}}$ for the integration over t. Noting that

$$\mathrm{i}t_+ - b_+ = \mathrm{i}t_+ - u_+ = (\lambda - \mu)\left(\frac{t_+}{\sqrt{2-\lambda^2}}\right) \tag{112}$$

we find that the factors $1/\sqrt{f''(t_c)}$ cancel, and the second term becomes the same as the first term. We obtain by adding the two saddle points, in the limit $\lambda \simeq \mu$,

$$\rho(\lambda) = \frac{1}{\pi N} \lim_{\mu\to\lambda} \frac{\sin\left[N(\lambda-\mu)\sqrt{2-\lambda^2}\right]}{\lambda - \mu} = \frac{1}{\pi}\sqrt{2-\lambda^2} \tag{113}$$

which is indeed the semi-circle law in the large-N limit for the density of states.

6. Ratio of two characteristic polynomials and two-point correlation function: preliminaries

We now consider the ratio of two characteristic polynomials,

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \left\langle \frac{\det(\lambda_1 - X)\det(\lambda_2 - X)}{\det(\mu_1 - X)\det(\mu_2 - X)} \right\rangle \tag{114}$$

for the GOE ensemble. We introduce again Grassmann variables θ_1, θ_2 and complex numbers z_1, z_2 to represent these determinants,

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \int \prod_{a=1}^{N}\prod_{\alpha=1}^{2} \mathrm{d}z^*{}_{\alpha a}\, \mathrm{d}z_{\alpha a}\, \mathrm{d}\bar{\theta}_{\alpha a}\, \mathrm{d}\theta_{\alpha a} \times \left\langle \mathrm{e}^{\mathrm{i}N[\bar{\theta}_{1a}(\lambda_1\delta_{ab} - X_{ab})\theta_{1b} + \bar{\theta}_{2a}(\lambda_2\delta_{ab} - X_{ab})\theta_{2b} + z^*_{1a}(\mu_1\delta_{ab} - X_{ab})z_{1b} + z^*_{2a}(\mu_2\delta_{ab} - X_{ab})z_{2b}]}\right\rangle. \tag{115}$$

We integrate out the random matrix X by (45). Noting that tr Y^2 and tr YY^T are now

$$\mathrm{tr}\, Y^2 = -(\bar\theta_\alpha\theta_\beta)(\bar\theta_\beta\theta_\alpha) + (z^*_\alpha z_\beta)(z^*_\beta z_\alpha) + 2(\bar\theta_\alpha z_\beta)(z^*_\beta\theta_\alpha) \tag{116}$$

$$\mathrm{tr}\, YY^T = (\bar\theta_\alpha\bar\theta_\beta)(\theta_\beta\theta_\alpha) + (z^*_\alpha z^*_\beta)(z_\beta z_\alpha) + 2(\bar\theta_\alpha z^*_\beta)(z_\beta\theta_\alpha) \tag{117}$$

where $\alpha, \beta = 1, 2$, and $(\bar\theta_\alpha\theta_\beta) = \sum_{a=1}^N \bar\theta_{\alpha a}\theta_{\beta a}$. As for the one-point correlation function, one introduces auxiliary fields $b_1, b_2, b'_1, b'_2, \ldots$, and uses the Gaussian integral representation of (48). Then, one integrates out $\bar\theta, \theta, z^*, z$. The result is then cast into the form

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \int [\mathrm{Sdet}Q]^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{Str}\, Q^2 + \mathrm{Str}\, Q\Lambda} \tag{118}$$

where Q is an 8×8 supermatrix,

$$Q = \begin{pmatrix} b_1 & -v_1 & -v_3 & -v_4 & \mathrm{i}\bar\eta_1 & \mathrm{i}\eta_2 & \mathrm{i}\bar\eta_3 & \mathrm{i}\eta_4 \\ -v^*_1 & b_1 & -v^*_4 & -v^*_3 & \mathrm{i}\bar\eta_2 & \mathrm{i}\eta_1 & \mathrm{i}\bar\eta_4 & \mathrm{i}\eta_3 \\ -v^*_3 & -v_4 & b_2 & -v_2 & \mathrm{i}\bar\eta_5 & \mathrm{i}\eta_6 & \mathrm{i}\bar\eta_7 & \mathrm{i}\eta_8 \\ -v^*_4 & -v_3 & -v^*_2 & b_2 & \mathrm{i}\bar\eta_6 & \mathrm{i}\eta_5 & \mathrm{i}\bar\eta_8 & \mathrm{i}\eta_7 \\ -\mathrm{i}\eta_1 & -\mathrm{i}\eta_2 & -\mathrm{i}\eta_5 & -\mathrm{i}\eta_6 & -\mathrm{i}b'_1 & 0 & -w_1 & -w_2 \\ \mathrm{i}\bar\eta_2 & \mathrm{i}\bar\eta_1 & \mathrm{i}\bar\eta_6 & \mathrm{i}\bar\eta_5 & 0 & -\mathrm{i}b'_1 & -w^*_2 & -w^*_1 \\ -\mathrm{i}\eta_3 & -\mathrm{i}\eta_4 & -\mathrm{i}\eta_7 & -\mathrm{i}\eta_8 & w^*_1 & w_2 & -\mathrm{i}b'_2 & 0 \\ \mathrm{i}\bar\eta_4 & \mathrm{i}\bar\eta_3 & \mathrm{i}\bar\eta_8 & \mathrm{i}\bar\eta_7 & w^*_2 & w_1 & 0 & -\mathrm{i}b'_2 \end{pmatrix}. \tag{119}$$

Again one needs to diagonalize Q by a supergroup transformation, and find the Jacobian for the representation in terms of eigenvalues. This yields a factor $(t_1 - t_2)^4$ for the (b', w) block matrices of Q, as discussed in a previous article [7]. The Jacobian for the Grassmannian part is again obtained by linearizing the group transformation near identity as in (87). Then, in terms of the eigenvalues of Q, we have

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \int \frac{[(\mathrm{i}t_1)(\mathrm{i}t_2)]^N}{(u_1u_2u_3u_4)^{N/2}} \frac{\prod_{\mathrm{i}<j}^4 |u_i - u_j|(t_1 - t_2)^4}{\prod_{i=1}^4 (\mathrm{i}t_1 - u_i)^2 \prod_{i=1}^4 (\mathrm{i}t_2 - u_i)^2}\, \mathrm{e}^{-N\sum t_i^2 - N\sum_j u_j^2} \times I\, \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\, \mathrm{d}u_4 \tag{120}$$

where I is the HIZ integral

$$I = \int \mathrm{d}g\, \mathrm{e}^{N\,\mathrm{Str}\, gQg^{-1}\Lambda}. \tag{121}$$

It is sufficient to take diagonal matrices $Q = \mathrm{diag}(u_1, u_2, u_3, u_4, \mathrm{i}t_1, \mathrm{i}t_1, \mathrm{i}t_2, \mathrm{i}t_2)$, and Λ: $\Lambda = (\mu_1, \mu_1, \mu_2, \mu_2, \lambda_1, \lambda_1, \lambda_2, \lambda_2)$. Note that the eigenvalues of Λ are doubly degenerate. This HIZ integral is again computed by taking the Laplacian with respect to matrix elements, a generalization of what appeared in our previous work [7].

The Jacobian J in this case is the generalization of (88), which is derived by (87),

$$J = \frac{\prod_{i<j}^4 |u_i - u_j|(t_1 - t_2)^4}{\prod_{i=1}^2 \prod_{\alpha=1}^4 (\mathrm{i}t_i - u_\alpha)^2}. \tag{122}$$

The Laplacian equation is

$$\frac{1}{2J}\sum_{i=1}^2 \frac{\partial}{\partial t_i} J \frac{\partial}{\partial t_i} I(Q) + \frac{1}{J}\sum_{\alpha=1}^4 \frac{\partial}{\partial u_\alpha} J \frac{\partial}{\partial u_\alpha} I(Q) = \epsilon I(Q). \tag{123}$$

The factor $1/2$ comes from the degeneracy of t. We set again $h = J^{1/2}$ and substitute $\psi = \chi/h$. Then, the same identity (94) eliminates the first derivatives of χ. Since the eigenvalues of Λ are degenerate, we factor out $\sqrt{(u_1-u_2)(u_3-u_4)}$, and have the following HIZ integral,

$$I = \frac{\prod_{i=1,2;\alpha=1,\dots,4}(\mathrm{i}t_i-u_\alpha)(\lambda_1-\mu_1)^2(\lambda_1-\mu_2)^2(\lambda_2-\mu_1)^2(\lambda_2-\mu_2)^2}{(t_1-t_2)^2\prod_{i<j}\sqrt{u_i-u_j}(\lambda_1-\lambda_2)^2(\mu_1-\mu_2)^2} \times \sqrt{(u_1-u_2)(u_3-u_4)}\,g\,\mathrm{e}^{-2\mathrm{i}Nt_1\lambda_1-2\mathrm{i}Nt_2\lambda_2+N(u_1+u_2)\mu_1+N(u_3+u_4)\mu_2} + (\text{perm.}) \tag{124}$$

where (perm.) means the sum over permutations of the λ_i and μ_i. The differential equation becomes

$$\begin{aligned}
&-2\mathrm{i}N\lambda_1\frac{\partial g}{\partial t_1} - 2\mathrm{i}N\lambda_2\frac{\partial g}{\partial t_2} + 2N\mu_1\frac{\partial g}{\partial u_1} + 2N\mu_1\frac{\partial g}{\partial u_2} + 2N\mu_2\frac{\partial g}{\partial u_3} + 2N\mu_2\frac{\partial g}{\partial u_4} + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2 g}{\partial t_i{}^2}\\
&\quad + \sum_{\alpha=1}^{4}\frac{\partial^2 g}{\partial u_\alpha{}^2} + \frac{1}{u_1-u_2}\left(\frac{\partial g}{\partial u_1}-\frac{\partial g}{\partial u_2}\right) + \frac{1}{u_3-u_4}\left(\frac{\partial g}{\partial u_3}-\frac{\partial g}{\partial u_4}\right)\\
&\quad + g\left[-\frac{2}{(t_1-t_2)^2} - \sum_{i,\alpha}\frac{1}{(\mathrm{i}t_i-u_\alpha)^2} + \frac{1}{2(u_1-u_3)^2} + \frac{1}{2(u_1-u_4)^2}\right.\\
&\quad \left. + \frac{1}{2(u_2-u_3)^2} + \frac{1}{2(u_2-u_4)^2}\right] = 0.
\end{aligned} \tag{125}$$

The solution to this equation may be obtained by expanding in $1/N$; it contains several factors

$$\begin{aligned}
g = &\left[1-\frac{\mathrm{i}}{Nt_{12}}\right]\left[1+\frac{1}{4N}\left(\frac{1}{u_{13}}+\frac{1}{u_{14}}+\frac{1}{u_{23}}+\frac{1}{u_{24}}\right)+\cdots\right]\left[1-\frac{1}{2N}\left(\sum_{i=1}^{2}\sum_{\alpha=1}^{4}\frac{1}{\tau_{i\alpha}}\right)\right.\\
&+\frac{1}{4N^2}\left(\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}\right)\left(\frac{1}{\tau_{13}}+\frac{1}{\tau_{14}}\right)+\left(\frac{1}{\tau_{21}}+\frac{1}{\tau_{22}}\right)\left(\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)\right.\\
&\left.+\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}+\frac{1}{\tau_{13}}+\frac{1}{\tau_{14}}\right)\left(\frac{1}{\tau_{21}}+\frac{1}{\tau_{22}}+\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)\right)\\
&-\frac{1}{8N^3}\left(\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}\right)\left(\frac{1}{\tau_{13}}+\frac{1}{\tau_{14}}\right)\left(\frac{1}{\tau_{21}}+\frac{1}{\tau_{22}}+\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)\right.\\
&\left.+\left(\frac{1}{\tau_{21}}+\frac{1}{\tau_{22}}\right)\left(\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}+\frac{1}{\tau_{13}}+\frac{1}{\tau_{14}}\right)\right)\\
&\left.+\frac{1}{16N^4}\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}\right)\left(\frac{1}{\tau_{13}}+\frac{1}{\tau_{14}}\right)\left(\frac{1}{\tau_{21}}+\frac{1}{\tau_{22}}\right)\left(\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)\right]
\end{aligned} \tag{126}$$

where we define

$$\begin{aligned}
&t_{12} = (t_1-t_2)(\lambda_1-\lambda_2) & &\\
&u_{13} = (u_1-u_3)(\mu_1-\mu_2) & &u_{14} = (u_1-u_4)(\mu_1-\mu_2)\\
&u_{23} = (u_2-u_3)(\mu_1-\mu_2) & &u_{24} = (u_2-u_4)(\mu_1-\mu_2)\\
&\tau_{11} = (\mathrm{i}t_1-u_1)(\lambda_1-\mu_1) & &\tau_{13} = (\mathrm{i}t_1-u_3)(\lambda_1-\mu_2)\\
&\tau_{12} = (\mathrm{i}t_1-u_2)(\lambda_1-\mu_1) & &\text{etc.}
\end{aligned} \tag{127}$$

Note that all these variables Nt, $N\tau$, Nu are of order one in Dyson's scaling limit. The series in $1/\tau$ terminate at order $1/N^4$. However, the second factor in (126) is an infinite series in powers of $1/N$, and the scaling limit requires a determination of the full series. Remarkably enough, a closed solution may be found in the scaling limit. This series turns out to be simply the HIZ integral for the GOE ensemble, and it will be discussed in the next section before returning to the two-point function.

7. Heat kernel equation for non-unitary Itzykson–Zuber integrals

Let us consider the integral

$$I = \int \mathrm{d}g\, \mathrm{e}^{N\,\mathrm{Tr}(gAg^{-1}B)} \tag{128}$$

in which g runs over the orthogonal group $O(2k)$, with the usual Haar measure of integration. We do consider group integration in this section, not supergroups. We shall also later generalize it to other values of β with, as usual, $\beta = 1, 2, 4$ for the GOE, GUE and GSE ensembles, respectively. The eigenvalues of A are denoted as $(u_1, \ldots, u_{2k})$, and the eigenvalues of B as $(\mu_1, \ldots, \mu_{2k})$. Considered as a function of the matrix A the integral I satisfies a Laplacian equation

$$\Delta_A I = \epsilon I \tag{129}$$

where Δ_A is the Laplacian with respect to the matrix elements of A, and

$$\epsilon = N^2\,\mathrm{Tr}\,B^2 = N^2 \sum_1^{2k} \mu_a^2. \tag{130}$$

Since I is a function of the eigenvalues u_i of A one may replace the Laplacian on the matrix elements by a differential operator on the eigenvalues

$$\Delta = \frac{1}{J} \sum_{i=1}^{2k} \frac{\partial}{\partial u_i} J \frac{\partial}{\partial u_i} \tag{131}$$

where the measure J is given by the absolute value of the Vandermonde determinant,

$$J = \prod_{i>j}^{2k} |u_i - u_j|. \tag{132}$$

Let us investigate the simplest case of a 2×2 matrix ($k = 1$), for which one obtains

$$\left[\frac{\partial^2}{\partial {u_1}^2} + \frac{\partial^2}{\partial {u_2}^2} + \frac{1}{u_1 - u_2}\frac{\partial}{\partial u_1} - \frac{1}{u_1 - u_2}\frac{\partial}{\partial u_2}\right] I = \epsilon I. \tag{133}$$

To eliminate the first-order derivatives, one substitutes

$$I = \frac{1}{\sqrt{u_1 - u_2}}\chi. \tag{134}$$

Then (129) becomes

$$\frac{\partial^2 \chi}{\partial {u_1}^2} + \frac{\partial^2 \chi}{\partial {u_2}^2} + \frac{1}{2}\frac{1}{(u_1 - u_2)^2}\chi = \epsilon\chi. \tag{135}$$

One then substitutes further

$$\chi = \mathrm{e}^{Nu_1\mu_1 + Nu_2\mu_2} f. \tag{136}$$

Noting that $\epsilon = N^2(\mu_1{}^2 + \mu_2{}^2)$, we obtain

$$2N\mu_1 \frac{\partial f}{\partial u_1} + 2N\mu_2 \frac{\partial f}{\partial u_2} + \frac{\partial^2 f}{\partial u_1{}^2} + \frac{\partial^2 f}{\partial u_2{}^2} + \frac{1}{2(u_1 - u_2)^2} f = 0. \tag{137}$$

From this differential equation one generates the expansion of the function f in powers of $1/N$

$$f = 1 + \frac{1}{4N(u_1 - u_2)(\mu_1 - \mu_2)} + \frac{9}{32N^2(u_1 - u_2)^2(\mu_1 - \mu_2)^2} + O\left(\frac{1}{N^3}\right). \tag{138}$$

This series is in fact the expansion of a modified Bessel function, as may be recognized on the differential equation. Indeed if we look for a solution f, which is a function of the single scaling variable

$$x = N(u_1 - u_2)(\mu_1 - \mu_2) \tag{139}$$

one sees that $f(x)$ should be the solution of the differential equation

$$f''(x) + f'(x) + \frac{1}{4x^2} f = 0. \tag{140}$$

Therefore the $k = 1$ solution is

$$I = \mathrm{e}^{\frac{N}{2}(u_1+u_2)(\mu_1+\mu_2)} I_0\left(\frac{N}{2}(u_1 - u_2)(\mu_1 - \mu_2)\right) \tag{141}$$

where $I_0(z)$ is a modified Bessel function, whose asymptotic expansion is

$$I_0(x) = \frac{\mathrm{e}^x}{\sqrt{2\pi x}}\left[1 + \frac{1}{8x} + \frac{9}{2(8x)^2} + \frac{225}{6(8x)^3} + \cdots\right] \tag{142}$$

in agreement with (138).

In the expansion (138), we have implicitly assumed that $(u_1 - u_2)$ is of order one in the Dyson limit, where $N(\mu_1 - \mu_2)$ is of order one. However, if $(u_1 - u_2) \sim O(1/N)$, the previous expansion is not valid and one has to perform a different expansion of the solution of the differential equation (137). Instead of (138) one expands

$$f = C\sqrt{|u_1 - u_2|}\left[1 - \frac{N}{2}(\mu_1 - \mu_2)(u_1 - u_2) + \cdots\right] \tag{143}$$

in order to be able to deal with the regime in which the difference of the two arguments u_1 and u_2 becomes of the order of $O(1/N)$. It corresponds, of course, to the small x-expansion of the modified Bessel function (142).

Let us now proceed to the $k = 2$ case, which is relevant to our problem. We have $J = \prod_{1\leqslant i<j\leqslant 4} |u_i - u_j|$, and consider the region $u_1 > u_2 > u_3 > u_4$. The differential equation for I reads

$$\sum_{i=1}^{4} \frac{\partial^2}{\partial u_i{}^2} I + \sum_{i=1}^{4}\sum_{j\neq i} \frac{1}{u_i - u_j}\frac{\partial}{\partial u_i} I = \epsilon I. \tag{144}$$

Substituting $I = \frac{1}{\sqrt{|\Delta(u)|}}\chi$, where $\Delta(u)$ is the Vandermonde determinant of the u_i $(i = 1, \ldots, 4)$, one has

$$\sum_{i=1}^{4} \frac{\partial^2}{\partial u_i{}^2}\chi + \frac{1}{2}\chi \sum_{i<j} \frac{1}{(u_i - u_j)^2} = \epsilon\chi. \tag{145}$$

Again one substitutes $\chi = \mathrm{e}^{Nu_1\mu_1+Nu_2\mu_2+Nu_3\mu_3+Nu_4\mu_4} f$, and finds

$$2N\sum_{i=1}^{4}\mu_i\frac{\partial f}{\partial u_i}+\sum_{i=1}^{4}\frac{\partial^2 f}{\partial {u_i}^2}+\frac{1}{2}\sum_{i<j}\frac{1}{(u_i-u_j)^2}f=0. \tag{146}$$

This equation has beautiful properties in the scaling limit of interest. Before discussing this property, let us write the generalization of this equation for arbitrary β ($\beta = 1, 2, 4$ are the standard values for GOE, GUE and GSE respectively), and for k variables instead of four:

$$2N\sum_{i=1}^{k}\mu_i\frac{\partial f}{\partial u_i}+\sum_{i=1}^{k}\frac{\partial^2 f}{\partial {u_i}^2}-y\sum_{i<j}\frac{1}{(u_i-u_j)^2}f=0 \tag{147}$$

in which y stands for

$$y=\beta\left(\frac{\beta}{2}-1\right). \tag{148}$$

It is a remarkable property that this equation has a solution which is a function of the scaling variables

$$\tau_{ij}=N(\mu_i-\mu_j)(u_i-u_j) \tag{149}$$

alone, and not separately of the u and μ. Given the origin of this equation, namely the HIZ integrals, it is clear that the solution involves only dimensionless products of the type $\mu_i \cdot u_j$, and that it is unchanged by a simultaneous permutation of u_i and u_j accompanied by the same permutation on $\mu_i \leftrightarrow \mu_j$. However, a direct proof or verification on the equation itself turns out to lead to very elaborate combinatorial identities if one tries, for instance, to expand the solution for large τ [19]. The lowest orders are simple, but the calculations become very tedious, and non-trivial, at higher orders. Let us reproduce here simply the first terms:

$$f=1-\sum_{i<j}\frac{y}{2\tau_{ij}}-\sum_{i<j}\frac{y}{2}\frac{1}{\tau_{ij}^2}+\frac{y^2}{8}\left(\sum_{i<j}\frac{1}{\tau_{ij}}\right)^2+\cdots. \tag{150}$$

This expansion shows our point: the coefficients are pure numbers, completely independent of the parameters μ, once the function is expressed in terms of the τ_{ij}.

This is a generalization of the expansion of the modified Bessel function (138); one recovers the simple Bessel limit if one lets all the u, except the first two, go to infinity.

As discussed above, all we need in the large-N limit is a solution for the degenerate case, $\mu_1 = \mu_2 = \mu$, and $\mu_3 = \mu_4 = \mu'$. To deal with this degenerate case, one writes f as

$$f=\sqrt{|(u_1-u_2)(u_3-u_4)|}\,g. \tag{151}$$

From (146), we have

$$\begin{aligned}
&2N\mu\frac{\partial g}{\partial u_1}+2N\mu\frac{\partial g}{\partial u_2}+2N\mu'\frac{\partial g}{\partial u_3}+2N\mu'\frac{\partial g}{\partial u_4}+\sum_{\alpha=1}^{4}\frac{\partial^2 g}{\partial {u_\alpha}^2}\\
&\quad+\frac{1}{u_1-u_2}\left(\frac{\partial g}{\partial u_1}-\frac{\partial g}{\partial u_2}\right)+\frac{1}{u_3-u_4}\left(\frac{\partial g}{\partial u_3}-\frac{\partial g}{\partial u_4}\right)\\
&\quad+\left[\frac{1}{2}\frac{1}{(u_1-u_3)^2}+\frac{1}{2}\frac{1}{(u_1-u_4)^2}+\frac{1}{2}\frac{1}{(u_2-u_3)^2}+\frac{1}{2}\frac{1}{(u_2-u_4)^2}\right]g=0.
\end{aligned} \tag{152}$$

This function g may then be obtained as an expansion in powers of $\frac{1}{N}$. Terms involving poles in $\frac{1}{u_1-u_2}$ and $\frac{1}{u_3-u_4}$ do not appear in the expression for g. We have

$$\begin{aligned} g = 1 &+ \frac{1}{4N(u_1-u_3)(\mu-\mu')} + \frac{1}{4N(u_1-u_4)(\mu-\mu')} + \frac{1}{4N(u_2-u_3)(\mu-\mu')} \\ &+ \frac{1}{4N(u_2-u_4)(\mu-\mu')} + \frac{9}{32N^2(\mu-\mu')^2} \\ &\times \left[\frac{1}{(u_1-u_3)^2} + \frac{1}{(u_1-u_4)^2} + \frac{1}{(u_2-u_3)^2} + \frac{1}{(u_2-u_4)^2}\right] \\ &+ \frac{1}{16N^2(\mu-\mu')^2}\left[\frac{1}{(u_1-u_3)(u_2-u_4)} + \frac{1}{(u_1-u_4)(u_2-u_3)}\right] \\ &+ \frac{3}{16N^2(\mu-\mu')^2}\left[\frac{1}{(u_1-u_3)(u_1-u_4)} + \frac{1}{(u_1-u_3)(u_2-u_3)}\right. \\ &\left. + \frac{1}{(u_1-u_4)(u_2-u_4)} + \frac{1}{(u_2-u_3)(u_2-u_4)}\right] + O\left(\frac{1}{N^3}\right). \end{aligned} \tag{153}$$

This expression has a complicated structure; one may write the various terms as diagrams, but the coefficients depend upon the topological character of those diagrams. For instance, for the coefficients at order $1/N^2$ the three types of diagrams have weights 9/32, 1/16 and 3/16. However, we find that in the large-N limit the expressions simplify.

In the large-N limit, if we concentrate on the saddle points $u_1 = u_2 = u_+$ and $u_3 = u_4 = u_-$, one obtains

$$g = 1 + \frac{1}{N(u_+-u_-)(\mu-\mu')} + \frac{2}{N^2(u_+-u_-)^2(\mu-\mu')^2} + O\left(\frac{1}{N^3}\right). \tag{154}$$

If we introduce the scaling variable

$$x = N(u_+-u_-)(\mu-\mu') \tag{155}$$

one finds that $g(x)$ satisfies the differential equation

$$\frac{\partial^2 g}{\partial x^2} + \left(1 - \frac{1}{x}\right)\frac{\partial g}{\partial x} + \frac{g}{x^2} = 0. \tag{156}$$

The expansion of g in powers of $1/x$ is

$$g = \sum_{n=0}^{\infty} \frac{n!}{x^n}. \tag{157}$$

It is interesting to generalize this formula to general k ($k > 4$). Denoting $v_j = u_{k+j}$ ($j = 1, \ldots, k$), one deals with the differential equation

$$\begin{aligned} &2N\mu\sum_{\alpha=1}^{k}\frac{\partial g}{\partial u_\alpha} + 2N\mu'\sum_{\alpha=1}^{k}\frac{\partial g}{\partial v_\alpha} + \sum_{\alpha=1}^{k}\frac{\partial^2 g}{\partial {u_\alpha}^2} + \sum_{\alpha=1}^{k}\frac{\partial^2 g}{\partial {v_\alpha}^2} + \sum_{\alpha<\beta}\frac{1}{u_\alpha-u_\beta}\left(\frac{\partial g}{\partial u_\alpha} - \frac{\partial g}{\partial u_\beta}\right) \\ &\qquad + \sum_{\alpha<\beta}\frac{1}{v_\alpha-v_\beta}\left(\frac{\partial g}{\partial v_\alpha} - \frac{\partial g}{\partial v_\beta}\right) + g\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left[\frac{1}{2}\frac{1}{(u_\alpha-v_\beta)^2}\right] = 0. \end{aligned} \tag{158}$$

For general k, the situation is similar. In the scaling limit, in which the leading saddle points involve only two distinct values of the form $u_a = u_+$ and $v_a = u_-$, one introduces the same scaling variable x (155), and obtains the differential equation

$$\frac{\partial^2 g}{\partial x^2} + \left(1 - \frac{k-1}{x}\right)\frac{\partial g}{\partial x} + \frac{k^2}{4x^2}g = 0. \tag{159}$$

An expansion in powers of $1/x$ of the solution follows easily:

$$g = 1 + \sum_{p=1}^{\infty} \frac{[k(k+2)(k+4)\cdots(k+2(p-1))]^2}{2^{2p}p!x^p}. \tag{160}$$

One recovers the previous results for $k = 1$ and $k = 2$ from this expression.

We have considered hereabove the GOE measure $J = \prod_{i>j}|u_i - u_j|$. Let us now go to arbitrary β, i.e. work with $J = \prod_{i>j}|u_i - u_j|^\beta$; the GUE integral corresponds to $\beta = 2$ and the GSE to $\beta = 4$. When the eigenvalues μ_i, which are set at the saddle points in the scaling limit, degenerate again into two groups, one equal to μ and the other one to μ', one finds through identical steps

$$2N\mu\sum_{\alpha=1}^{k}\frac{\partial g}{\partial u_\alpha} + 2N\mu'\sum_{\alpha=1}^{k}\frac{\partial g}{\partial v_\alpha} + \sum_{\alpha=1}^{k}\frac{\partial^2 g}{\partial {u_\alpha}^2} + \sum_{\alpha=1}^{k}\frac{\partial^2 g}{\partial {v_\alpha}^2} + \beta\sum_{\alpha<\beta}\frac{1}{u_\alpha - u_\beta}\left(\frac{\partial g}{\partial u_\alpha} - \frac{\partial g}{\partial u_\beta}\right)$$
$$+ \beta\sum_{\alpha<\beta}\frac{1}{v_\alpha - v_\beta}\left(\frac{\partial g}{\partial v_\alpha} - \frac{\partial g}{\partial v_\beta}\right) + \frac{\beta(2-\beta)}{2}\left[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\frac{1}{(u_\alpha - v_\beta)^2}\right]g = 0. \tag{161}$$

In the scaling limit, in which the u and the v approach respectively u_+ and u_-, one introduces again the scaling variable (155) and obtains the differential equation

$$\frac{\partial^2 g}{\partial x^2} + \left(1 - \frac{k-1}{x}\right)\frac{\partial g}{\partial x} + \frac{k^2\beta(2-\beta)}{4x^2}g = 0 \tag{162}$$

which generalizes the previous GOE expression for $\beta = 1$. The expansion of the general β, arbitrary k solution, in powers of $1/x$ follows

$$g = 1 + \sum_{p=1}^{\infty}\frac{(\beta k)((2-\beta)k)(\beta k+2)(2+k(2-\beta))\cdots(2(p-1)+k(2-\beta))}{2^{2p}p!x^p}. \tag{163}$$

In the unitary case, $\beta = 2$, the solution reduces to the first term $g = 1$: this is of course the well-known Itzykson–Zuber result which is semiclassically exact.

For $\beta = 4$, and $k = 1$, we also see that the asymptotic expansion (163) stops at first order, since the terms with $p > 1$ vanish in the series (163), a fact that we had already found and used in (126) for integrating over the t_i. For $\beta = 4$, and $k = 2$, it stops at third order. The fact that the Itzykson–Zuber integral for this scaling limit of the GSE is semiclassical with only a *finite* number of corrections had already been discussed and used in [7].

Therefore one sees that, in the scaling limit of interest, in which we deal with an Itzykson–Zuber integral for a degenerate case, there is a remarkably simple expression for arbitrary β and arbitrary dimension k of the group integration. The expression is either semiclassically exact for the GUE, corrected by a finite number of terms for the GSE, or an infinite, but explicit, series for the GOE. The fact that k, the dimension of the integral, appears as a parameter will allow us later to continue in k and use the replica method in the k that goes to the zero limit.

Returning to the GOE, let us note that if we had included an i in the exponent of the Itzykson–Zuber integral in (128), and multiplied by the factor e^{ix}/x, we would have obtained for $k = 2$

$$\mathrm{Re}\left[\frac{e^{ix}}{x}g(ix)\right] = \frac{\cos x}{x} + \frac{\sin x}{x^2} - \frac{2\cos x}{x^3} - \frac{6\sin x}{x^4} + \cdots$$
$$= \sum_{k=1}^{\infty}\frac{(k-1)!\sin\left(x + \frac{\pi}{2}k\right)(-1)^{(k-1)}}{x^k} \tag{164}$$

which is the large x asymptotic expansion of the integral

$$I = \int_x^\infty \frac{\sin z}{z}\,\mathrm{d}z. \tag{165}$$

This integral appears in the scaling limit of the resolvent–resolvent correlation function of the GOE ensemble.

8. Two-point correlation function

We are now in a position to return to the two-point correlation function of the GOE, which will be deduced from the ratio of products of two characteristic polynomials. Indeed, one may obtain the resolvent–resolvent correlation function

$$\rho(\lambda_1, \lambda_2) = \frac{1}{\pi^2 N^2} \frac{\partial}{\partial\lambda_2} \frac{\partial}{\partial\lambda_1} \left\langle \frac{\det(\lambda_1 - X)\det(\lambda_2 - X)}{\det(\mu_1 - X)\det(\mu_2 - X)} \right\rangle \Bigg|_{\mu_1=\lambda_1,\mu_2=\lambda_2} \tag{166}$$

whose double discontinuity gives the two-level correlation function. For this correlation function, the relevant terms in the series g discussed in the previous section are those which involve a factor $\left(\frac{1}{\tau_{11}}+\frac{1}{\tau_{12}}\right)\left(\frac{1}{\tau_{23}}+\frac{1}{\tau_{24}}\right)$ since this factor yields pole terms of the form $\frac{1}{(\lambda_1-\mu_1)(\lambda_2-\mu_2)}$. Therefore, the relevant part of g, after taking derivatives and focusing on terms with poles at $\mu_i = \lambda_i$, is

$$\begin{aligned} g_L = {} & \frac{1}{4N^2} g_t g_u \left(\frac{1}{\mathrm{i}t_1 - u_1} + \frac{1}{\mathrm{i}t_1 - u_2}\right)\left(\frac{1}{\mathrm{i}t_2 - u_3} + \frac{1}{\mathrm{i}t_2 - u_4}\right) \frac{1}{(\lambda_1 - \mu_1)(\lambda_2 - \mu_2)} \\ & \times \left(1 - \frac{1}{2N}\left(\frac{1}{\mathrm{i}t_1 - u_3} + \frac{1}{\mathrm{i}t_1 - u_4}\right)\frac{1}{\lambda_1 - \mu_2}\right) \\ & \times \left(1 - \frac{1}{2N}\left(\frac{1}{\mathrm{i}t_2 - u_1} + \frac{1}{\mathrm{i}t_2 - u_2}\right)\frac{1}{\lambda_2 - \mu_1}\right) \end{aligned} \tag{167}$$

with

$$g_t = 1 - \frac{\mathrm{i}}{N t_{12}}. \tag{168}$$

In the large-N limit, we have three types of saddle points: (i) $\left(t_1^+, t_2^+, u_1^+, u_2^+, u_3^+, u_4^+\right)$ or $\left(t_1^-, t_2^-, u_1^-, u_2^-, u_3^-, u_4^-\right)$, (ii) $\left(t_1^+, t_2^-, u_1^+, u_2^+, u_3^-, u_4^-\right)$, $\left(t_1^+, t_2^-, u_1^-, u_2^-, u_3^+, u_4^+\right)$, $\left(t_1^-, t_2^+, u_1^+, u_2^+, u_3^-, u_4^-\right)$, or $\left(t_1^-, t_2^+, u_1^-, u_2^-, u_3^+, u_4^+\right)$, (iii) $\left(t_1^+, t_2^-, u_1^+, u_2^-, u_3^+, u_4^-\right)$, and similar combinations.

In those expressions the saddle points are given by

$$t^\pm = \frac{-\mathrm{i}\lambda \pm \sqrt{2-\lambda^2}}{2} \tag{169}$$

$$u^\pm = \frac{\mu \pm \mathrm{i}\sqrt{2-\mu^2}}{2} \tag{170}$$

with $\lambda_1, \lambda_2, \mu_1, \mu_2$.

For the saddle points of type (i), g_u is simply

$$g_u = \sqrt{|(u_1 - u_3)(u_1 - u_4)(u_2 - u_3)(u_2 - u_4)|}(\mu_1 - \mu_2)^2. \tag{171}$$

Thus we have, as the partial contribution of this saddle point to the correlation function, $\rho^{(i)}(\lambda_1, \lambda_2)$ from (124) and (167),

$$\rho^{(i)}(\lambda_1,\lambda_2) = \mathrm{Im} \lim_{\mu_1\to\lambda_1,\mu_2\to\lambda_2} \frac{\partial^2}{\partial\lambda_2\partial\lambda_1} \int \mathrm{d}t\,\mathrm{d}u \frac{(\mathrm{i}t_1)^N(\mathrm{i}t_2)^N}{(u_1u_2u_3u_4)^{\frac{N}{2}}} \frac{(t_1-t_2)^2\prod|u_i-u_j|}{\prod_{j=1}^2\prod_{\alpha=1}^4(\mathrm{i}t_j-u_\alpha)}$$
$$\times(\lambda_1-\mu_1)(\lambda_1-\mu_2)^2(\lambda_2-\mu_1)^2(\lambda_2-\mu_2)\left(\frac{1}{\mathrm{i}t_1-u_1}+\frac{1}{\mathrm{i}t_1-u_2}\right)$$
$$\times\left(\frac{1}{\mathrm{i}t_2-u_3}+\frac{1}{\mathrm{i}t_2-u_4}\right)\left[1-\frac{\mathrm{i}}{N(t_1-t_2)(\lambda_1-\lambda_2)}\right]$$
$$\times\left[1-\frac{1}{2N}\left(\frac{1}{\mathrm{i}t_1-u_3}+\frac{1}{\mathrm{i}t_1-u_4}\right)\frac{1}{\lambda_1-\mu_2}\right]$$
$$\times\left[1-\frac{1}{2N}\left(\frac{1}{\mathrm{i}t_2-u_1}+\frac{1}{\mathrm{i}t_2-u_2}\right)\frac{1}{\lambda_2-\mu_1}\right]\frac{1}{(\lambda_1-\lambda_2)^2}$$
$$\times\,\mathrm{e}^{-N(t_1^2+t_2^2)-\frac{N}{2}(u_1^2+u_2^2+u_3^2+u_4^2)+N(u_1+u_2)\mu_1+N(u_3+u_4)\mu_2-2\mathrm{i}Nt_1\lambda_1-2\mathrm{i}Nt_2\lambda_2}$$
$$+\,\mathrm{perm.}(\lambda_1\leftrightarrow\lambda_2,\mu_1\leftrightarrow\mu_2) \tag{172}$$

from which follows

$$\rho^{(i)}(\lambda_1,\lambda_2) = \mathrm{Im} \lim_{\mu_1\to\lambda_1,\mu_2\to\lambda_2} \frac{\partial^2}{\partial\lambda_2\partial\lambda_1} \int \mathrm{d}t\,\mathrm{d}u \frac{(\mathrm{i}t_1)^N(\mathrm{i}t_2)^N}{(u_1u_2u_3u_4)^{\frac{N}{2}}} \frac{(t_1-t_2)^2\prod|u_i-u_j|}{\prod_{j=1}^2\prod_{\alpha=1}^4(\mathrm{i}t_j-u_\alpha)}$$
$$\times(\lambda_1-\mu_1)(\lambda_1-\mu_2)(\lambda_2-\mu_1)(\lambda_2-\mu_2)\left(\frac{1}{\mathrm{i}t_1-u_1}+\frac{1}{\mathrm{i}t_1-u_2}\right)$$
$$\times\left(\frac{1}{\mathrm{i}t_2-u_3}+\frac{1}{\mathrm{i}t_2-u_4}\right)\frac{1}{4N^2}\left(\frac{1}{\mathrm{i}t_1-u_3}+\frac{1}{\mathrm{i}t_1-u_4}\right)\left(\frac{1}{\mathrm{i}t_2-u_1}+\frac{1}{\mathrm{i}t_2-u_2}\right)$$
$$\times\frac{1}{(\lambda_1-\lambda_2)^2}\,\mathrm{e}^{-N(t_1^2+t_2^2)-\frac{N}{2}(u_1^2+u_2^2+u_3^2+u_4^2)+N(u_1+u_2)\mu_1+N(u_3+u_4)\mu_2-2\mathrm{i}Nt_1\lambda_1-2\mathrm{i}Nt_2\lambda_2}. \tag{173}$$

For those saddle points of type (i), the difference $t_1 - t_2$ is of order $1/N$. Then, cancelling terms between the numerator and the denominator, one obtains

$$\rho^{(i)}(\lambda_1,\lambda_2) \simeq \mathrm{Im}\int \mathrm{d}t\,\mathrm{d}u \frac{(\mathrm{i}t_1)^N(\mathrm{i}t_2)^N}{(u_1u_2u_3u_4)^{\frac{N}{2}}}\frac{|u_1-u_2||u_3-u_4|}{(\mathrm{i}t_1-u_1)(\mathrm{i}t_1-u_2)}\left(\frac{1}{\mathrm{i}t_1-u_1}+\frac{1}{\mathrm{i}t_1-u_2}\right)$$
$$\times\frac{1}{(\mathrm{i}t_2-u_3)(\mathrm{i}t_2-u_4)}\left(\frac{1}{\mathrm{i}t_2-u_3}+\frac{1}{\mathrm{i}t_2-u_4}\right)\frac{1}{N^2}$$
$$\times\,\mathrm{e}^{-N(t_1^2+t_2^2)-2\mathrm{i}Nt_1\lambda_1-2\mathrm{i}Nt_2\lambda_2-\frac{N}{2}(u_1^2+u_2^2+u_3^2+u_4^2)+N(u_1+u_2)\lambda_1+N(u_3+u_4)\lambda_2} \tag{174}$$

which is just a product of density of states as given in equation (101). Therefore, by considering the saddle point of type (i) $\left(t_1^+, t_2^+, u_1^+, u_2^+, u_3^+, u_4^+\right)$ and $\left(t_1^-, t_2^-, u_1^-, u_2^-, u_3^-, u_4^-\right)$, we obtain

$$\rho^{(i)}(\lambda_1,\lambda_2) = \rho(\lambda_1)\rho(\lambda_2). \tag{175}$$

We now return to the integral (173), and consider a saddle point of type (ii); one thereby obtains the sin-kernel squared, since now terms such as $(\mathrm{i}t_1 - u_1)$ are replaced by the density of states $\rho(\lambda)$. The exchange of pairs between (u_1, u_2) and (u_3, u_4) in the denominator gives

$$\rho^{(ii)}(\lambda_1,\lambda_2) = -\frac{\sin^2(\pi N\rho(\lambda)(\lambda_1-\lambda_2))}{N^2\pi^2(\lambda_1-\lambda_2)^2}. \tag{176}$$

This term adds to (175), and by adding both terms, one obtains the expected vanishing contribution in the limit $\lambda_1 \to \lambda_2$.

Now, we take into account the remaining terms, g_t and g_u. In the case (ii) $u_1 - u_3$ is now proportional to the density of states ρ, which is of order one. Then all terms in g_u are of the same order in the Dyson limit, and we have to return to the discussion of the previous section. For the saddle point of type (iii), such as $\left(t_1^+, t_2^-, u_1^+, u_2^+, u_3^-, u_4^-\right)$, one has

$$\rho(\lambda_1, \lambda_2) = \mathrm{Im} \int \mathrm{d}t\, \mathrm{d}u \frac{(\mathrm{i}t_1)^N (\mathrm{i}t_2)^N}{(u_1 u_2 u_3 u_4)^{\frac{N}{2}}} \frac{2\mathrm{i}t_1 - u_3 - u_4}{(\mathrm{i}t_1 - u_3)^2 (\mathrm{i}t_1 - u_4)^2} \frac{2\mathrm{i}t_2 - u_3 - u_4}{(\mathrm{i}t_2 - u_1)^2 (\mathrm{i}t_2 - u_2)^2}$$
$$\times \left[1 - \frac{\mathrm{i}}{N\pi\rho(\lambda_1 - \lambda_2)}\right]\left[1 + \frac{\mathrm{i}}{4N\pi\rho(\lambda_1 - \lambda_2)} + \cdots\right]$$
$$\times\, \mathrm{e}^{-N(t_1^2+t_2^2) - 2\mathrm{i}Nt_1\lambda_1 - 2\mathrm{i}Nt_2\lambda_2 - \frac{N}{2}(u_1^2+u_2^2+u_3^2+u_4^2) + N(u_1+u_2)\lambda_1 + N(u_3+u_4)\lambda_2} \tag{177}$$

in which we have replaced various terms by $\rho(\lambda)$ and $(\lambda_1 - \lambda_2)$ by using their saddle-point values. We note that $t_1^+ - t_2^- = \sqrt{2 - \lambda^2} = \pi\rho(\lambda)$. One introduces the same integration variables b and ρ already used in (67), and integrates over ρ_1 and ρ_2 as in (109). Then, there is a part of the integrand which has a double pole in $\frac{1}{(\lambda_1-\lambda_2)^2}$ in the large-N limit. The imaginary parts are then taken for λ_1 and λ_2 independently. Thus one obtains for g_t, after multiplication by a factor $\frac{1}{N\rho(\lambda)(\lambda_1-\lambda_2)}$,

$$\mathrm{Im} \frac{1}{N\pi\rho(\lambda_1 - \lambda_2)} \left[\mathrm{i} + \frac{1}{N\pi\rho(\lambda_1 - \lambda_2)}\right] \mathrm{e}^{-\mathrm{i}N\pi\rho(\lambda_1 - \lambda_2)}$$
$$= \frac{\cos(\pi N\rho(\lambda_1 - \lambda_2))}{N\rho(\lambda_1 - \lambda_2)} - \frac{\sin(\pi N\rho(\lambda_1 - \lambda_2))}{N^2\pi^2\rho^2(\lambda_1 - \lambda_2)^2}$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) \tag{178}$$

where x stands for $x = N\pi\rho(\lambda_1 - \lambda_2)$. Next we return to the Itzykson–Zuber factor g_u of (153). After multiplication of g_u by $\frac{1}{N\rho(\lambda)(\lambda_1-\lambda_2)}$, one obtains

$$\mathrm{Im}\left[\frac{\mathrm{i}g_u}{N\rho(\lambda)(\lambda_1 - \lambda_2)}\, \mathrm{e}^{N(u_1^- + u_2^-)\lambda_1 + N(u_3^+ + u_4^+)\lambda_2}\right]$$
$$= \frac{\cos x}{x} + \frac{\sin x}{x^2} - 2\frac{\cos x}{x^3} - 6\frac{\sin x}{x^4} + \cdots$$
$$= \int_x^\infty \frac{\sin z}{z}\, \mathrm{d}z. \tag{179}$$

Therefore, we have by multiplying these two factors,

$$\rho(\lambda_1, \lambda_2) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) \int_x^\infty \frac{\sin z}{z}\, \mathrm{d}z. \tag{180}$$

The third type of saddle point (iii), for instance $\left(t_1^+, t_2^-, u_1^+, u_2^-, u_3^+, u_4^-\right)$, does not yield any imaginary part, and may thus be dropped in this GOE calculation. However for GSE, they do contribute as well.

Adding the type (i) (175) and (ii) (176) results, one obtains the two-point correlation function for the GOE in the large-N limit,

$$\rho(\lambda_1, \lambda_2) = \rho^2(\lambda)\left[1 - \left(\frac{\sin x}{x}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) \int_x^\infty \frac{\sin z}{z}\, \mathrm{d}z\right] \tag{181}$$

where $x = \pi N\rho(\lambda)(\lambda_1 - \lambda_2)$. Comment: this result is of course well known [3]; it was obtained long ago through the technique of skew-orthogonal polynomials. The point of the long derivation presented here, through generalized Itzykson–Zuber integrals, is that it can be repeated in other cases, such as non-invariant measures involving an external source, for which the standard method does not apply.

9. Zero-replica limit

Up to now we have computed averages of ratios of characteristic polynomials defined in (1). The usual two-point correlation function of the resolvent operator is obtained from this ratio by differentiation, as shown in (166). Instead of such ratios one may use an alternative 'replica' method, as follows.

Consider the correlation function

$$F_{2k}(\lambda_1, \lambda_2) = \langle [\det(\lambda_1 - X)]^k [\det(\lambda_2 - X)]^k \rangle. \tag{182}$$

Since $[\det(\lambda - X)]^k = \exp[k \operatorname{tr} \log(\lambda - X)]$, one may recover the correlation functions of the resolvent, by letting the replica number k go to zero : $k \to 0$ since

$$\lim_{k\to 0} \frac{1}{k^2} \frac{\partial^2}{\partial\lambda_1 \partial\lambda_2} F_{2k}(\lambda_1, \lambda_2) = \left\langle \mathrm{Tr} \frac{1}{\lambda_1 - X} \mathrm{Tr} \frac{1}{\lambda_2 - X} \right\rangle. \tag{183}$$

Let us first discuss the zero-replica limit (182) for the GUE, and show that the well-known exact expression for the two-point correlation function of the resolvent is correctly recovered. There were discussions of the validity of the replica limit in [26].

In an earlier work [4], the universality of F_{2k} in the Dyson short-distance limit was proven and its expression was found to be (for a probability measure proportional to $\exp(-N \operatorname{Tr} V(X))$)

$$\begin{aligned} F_{2k}(\lambda_1, \lambda_2) = {} & \mathrm{e}^{\frac{Nk}{2}(V(\lambda_1)+V(\lambda_2))}\, \mathrm{e}^{-Nk} \frac{1}{k!} (2\pi\rho(\lambda))^{k^2} \\ & \times \oint \prod_1^k \frac{\mathrm{d}u_\alpha}{2\pi} \exp\left(-\mathrm{i}\sum_{\alpha=1}^k u_\alpha\right) \frac{\Delta^2(u_1, \ldots, u_k)}{\prod_{\alpha=1}^k (u_\alpha - x)^k (u_\alpha + x)^k} \end{aligned} \tag{184}$$

a special case of the general formula derived in [4] for different λ_α:

$$\begin{aligned} F_{2k}(\lambda_1, \ldots, \lambda_{2k}) = {} & \left\langle \prod_{\alpha=1}^{2k} \det(\lambda_\alpha - X) \right\rangle = \mathrm{e}^{\frac{N}{2}\sum_{i=1}^{2k} V(\lambda_i)}\, \mathrm{e}^{-Nk} \frac{1}{k!} (2\pi\rho(\lambda))^{k^2} \\ & \times \oint \prod_1^k \frac{\mathrm{d}u_\alpha}{2\pi} \exp\left(-\mathrm{i}\sum_{\alpha=1}^k u_\alpha\right) \frac{\Delta^2(u_1, \ldots, u_k)}{\prod_{\alpha=1}^k \prod_{l=1}^{2k} (u_\alpha - x_l)} \end{aligned} \tag{185}$$

in which one has used the scaling variables $x = N\pi\rho(\lambda)(\lambda_1 - \lambda_2)$, $x_l = 2\pi N\rho(\lambda)(\lambda_l - \lambda)$ and $\lambda = (\lambda_1 + \lambda_2)/2$. This contour integral is a compact representation of the sum over the $(2k)!/(k!)^2$ saddle points which govern the Dyson limit, and it is particularly useful for degenerate cases in which some of the λ_i are equal. Indeed, in those cases, subtle corrections are present in the sum over saddle points, which are not easy to handle if one lets the λ approach each other too soon. The poles due to the degeneracy of the λ_i have to be cancelled by the sum over permutations over all possible saddle points. Formulae (184) and (185) do give a correct answer for the degenerate cases of equal x_l.

The calculation is done as follows. One first shifts $u_\alpha \to u_\alpha = v_\alpha + x$ and drops all factors in F_{2k}, which approach one in the zero-replica limit. Then we have to deal with the integral

$$I = \oint \prod_1^k \frac{\mathrm{d}v_\alpha}{2\pi} \exp\left(-\mathrm{i}kx - \mathrm{i}\sum v_l\right) \frac{\Delta^2(v_1, \ldots, v_k)}{\prod_{1\leqslant l\leqslant k} v_l^{2k} (1 + 2x/v_l)^k} \tag{186}$$

which vanishes if one sets $k = 0$ in the integrand. The zero-k limit is slightly tricky since the number of integrations is also equal to k, and one has to find explicit answers for the integrals before approaching the required limit. We are interested here in terms of order k^2 for small k. Thus one expands the denominator in powers of x,

$$\frac{1}{(1+2x/v)^k} = 1 - k\sum_1^\infty \frac{(-2x)^p}{pv^p} + O(k^2). \tag{187}$$

Since the one gives a vanishing contour integral, up to order k^2, it is sufficient to expand one of the factors $(1+2x/v)^{-k}$ to order k, since there are k ways of singling out a particular v_α. If one expanded both $(1+x/v_\alpha)^{-2k}$ and $(1+x/v_\beta)^{-2k}$, the number of choices $k(k-1)/2$, together with the two powers of k, would give a term proportional to k^3.

Then, in the zero-replica limit, it is sufficient to examine the integral

$$I^{(1)}(x) = k\,\mathrm{e}^{-\mathrm{i}kx}\oint\prod\frac{\mathrm{d}v_\alpha}{2\pi}\,\mathrm{e}^{-\mathrm{i}\sum v_\alpha}\frac{\Delta^2(v)}{(1+2x/v_1)^k\prod_1^k v_l^{2k}}. \tag{188}$$

This integral makes it clear that there are oscillatory terms of the form $\mathrm{e}^{2\mathrm{i}x}$, the contribution of the pole $v_1 = -2x$, and a non-oscillating term which comes from the pole $v_1 = 0$. We will show indeed there are such terms and they are $\left(\frac{\sin x}{x}\right)^2$. One then expands $\mathrm{e}^{-\mathrm{i}kx}(v_1+2x)^{-k}$ in $I^{(1)}(x)$ in powers of x. This generates the following integrals:

$$\begin{aligned}\gamma_k^{(p)} &= \oint\frac{\mathrm{d}v}{2\pi}\frac{\Delta^2(v)\,\mathrm{e}^{-\mathrm{i}\sum v_\alpha}}{v_1^{2k+p}v_2^{2k}\cdots v_k^{2k}} = (-\mathrm{i})^p\prod_{l=0}^{k-1}\frac{l!}{(k+l)!}\\ &\quad\times\frac{(2+p)(4+p)\cdots((2k-2)+p)(p-3)(p-5)\cdots(p-(2k-1))}{(2k-1+p)!}\\ &= (-\mathrm{i})^p\prod_{l=0}^{k-1}\frac{l!}{(k+l)!}\frac{2\Gamma\left(2-\frac{p}{2}\right)\Gamma\left(k+\frac{p}{2}\right)\Gamma(2k-p)}{\Gamma(2k+p)\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(k-\frac{p}{2}\right)\Gamma(3-p)}.\end{aligned} \tag{189}$$

Note that when p is an odd integer, $\gamma_k^{(p)}$ does not contribute to $I^{(1)}(x)$, since one has to take a real part. When $p=0$, $\gamma_k^{(0)}$ coincides with a well-known universal number, related to the moments of the Riemann zeta function. In the zero-replica limit of $k\to 0$, we have found in an earlier work [5] that

$$\lim_{k\to 0}\prod_{l=0}^{k-1}\frac{l!}{(k+l)!} = 1 + k^2(1+c) + O(k^3) \tag{190}$$

where c is Euler's constant. Therefore we have

$$\lim_{k\to 0}\gamma_k^{(p)} = \frac{1}{(1-p)\Gamma(p+1)}(-(-\mathrm{i})^p). \tag{191}$$

By expanding $\frac{1}{(v_1+2x)^k}$ in powers of x, we obtain

$$I^{(1)}(x) = \sum_{p=\mathrm{even}}^{\infty}(-\mathrm{i})^p\left(\frac{k^2}{p}\right)(2x)^p\frac{1}{(1-p)\Gamma(p+1)}. \tag{192}$$

Thus the second derivative of $I^{(1)}$ with respect to x is finally given by

$$\frac{\partial^2 I^{(1)}}{\partial x^2} = -k^2\,\mathrm{Re}\sum_{p=\mathrm{even}}^{\infty}(-\mathrm{i})^p\frac{2^p}{p!}x^{p-2} = -k^2\frac{\mathrm{e}^{2\mathrm{i}x}+\mathrm{e}^{-2\mathrm{i}x}-2}{2x^2} = 2k^2\left(\frac{\sin x}{x}\right)^2. \tag{193}$$

Finally there is an additional constant term which comes from the second derivative of $\mathrm{e}^{-\mathrm{i}kx}$ in I, which was neglected in $I^{(1)}$. Therefore, adding this constant, we obtain the well-known two-point correlation function of the GUE by this replica method,

$$\rho(\lambda_1,\lambda_2) = \rho^2(\lambda)\left[1-\left(\frac{\sin x}{x}\right)^2\right]. \tag{194}$$

We may now proceed to the GOE case. The Itzykson–Zuber integral for the GOE case has been discussed in the previous section, when we dealt with the ratio of characteristic polynomials. We now consider the following moment:

$$I = \left\langle \frac{1}{[\det(\lambda_1 - X)]^k [\det(\lambda_2 - X)]^k} \right\rangle. \tag{195}$$

Again the zero-replica limit will be used to obtain the two-point correlation function of the resolvent operator for the GOE.

Indeed, here we deal again with the Itzykson–Zuber integral for a degenerate source, with only two distinct eigenvalues, λ_1 and λ_2, both k-times degenerate. In such cases the heat kernel satisfies the differential equation (159). The asymptotic solution of the solution in the scaling limit is given by (160),

$$g = 1 + \sum_{p=1}^{\infty} \frac{[k(k+2)(k+4)\cdots(k+2(p-1))]^2}{2^{2p} p! x^p}. \tag{196}$$

In the zero-replica limit $k \to 0$, we need to keep only the terms proportional to k^2. To order k^2, we have from (196)

$$g = 1 + \frac{k^2}{4}\left(\sum_p \frac{(p-1)!}{p x^p}\right). \tag{197}$$

Then taking the second derivative of g with respect to x, we have a factor $(p! + (p-1)!)/x^{p-2}$, which is precisely what one obtains from the asymptotic expansion of the standard two-point correlation function of the GOE ensemble,

$$\rho_2(x) = \rho^2(\lambda)\left[1 - \left(\frac{\sin x}{x}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)\int_x^{\infty} \frac{\sin z}{z}\,\mathrm{d}z\right]. \tag{198}$$

Indeed taking two derivatives with respect to x of the second and third terms:

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_x^{\infty} \frac{\sin z}{z}\,\mathrm{d}z\right)^2 = \left(\frac{\sin x}{x}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)\int_x^{\infty}\frac{\sin z}{z}\,\mathrm{d}z. \tag{199}$$

Next let us compare our expression of g with the factor $\left(\int_x^{\infty} \frac{\sin z}{z}\,\mathrm{d}z\right)^2$. From the asymptotic expansion of the integral

$$\int_x^{\infty} \frac{\sin z}{z}\,\mathrm{d}z = \operatorname{Im} \mathrm{e}^{\mathrm{i}x} \sum_{p\geqslant 0} \frac{p!}{2x^{p+1}}(-\mathrm{i})^p \tag{200}$$

one obtains for the square of this quantity

$$\left(\int_x^{\infty} \frac{\sin z}{z}\,\mathrm{d}z\right)^2 = \frac{1}{2}\sum_{p,p'} \frac{p!p'!}{x^{p+p'+2}}(-\mathrm{i})^p(\mathrm{i})^{p'} - \frac{1}{4}\mathrm{e}^{2\mathrm{i}x}\left(\sum \frac{p!}{2x^{p+1}}(-\mathrm{i})^p\right)^2$$
$$- \frac{1}{4}\mathrm{e}^{-2\mathrm{i}x}\left(\sum \frac{p!}{2x^{p+1}}(\mathrm{i})^p\right)^2. \tag{201}$$

Indeed our expansion (197) for g in the zero-replica limit agrees with the first term, the non-oscillating term of the above equation, since

$$\sum_{p,p'} \frac{p!p'!}{x^{p+p'+2}}(-\mathrm{i})^p(\mathrm{i})^{p'} = \int_0^{\infty} \frac{\mathrm{d}t}{t}\epsilon^{-t}\log\left(1 + \frac{t^2}{x^2}\right) = \sum_{p=0}(-1)^p \frac{(2p+2)!}{(p+1)x^{2p+2}}. \tag{202}$$

The Itzykson–Zuber integral which led to (196) has been derived by a saddle-point method, choosing $u_j = u_+$ $(j = 1, \ldots, k)$ and $u_l = u_-$ $(l = k; 1, \ldots, 2k)$. This gave a factor $\mathrm{e}^{\mathrm{i}kx}$

which reduces to unity in the limit $k \to 0$. Those saddle points contribute to the non-oscillating terms. However, there are other saddle points that one needs to consider in order to recover the oscillating part. This is analogous to a phenomenon recently analysed by Kamenev and Mézard [26]. As for the GUE, the zero-replica limit requires only the $e^{\pm 2ix}$ oscillating terms. Those terms may be obtained through saddle points of the following type: one divides the u_i ($i = 1, \ldots, 2k$) into two groups $(u_1, u_2, \ldots, u_k)$ and $(u_{k+1}, u_{k+2}, \ldots, u_{2k})$. One chooses one u_i to be a u_- in the first group, and the others u_+; similarly one u_+ in the second group, and the others are u_-. For instance, in the first group, $u_1 = u_+, u_2 = u_-, u_3 = u_+, \ldots, u_k = u_+$, and for the second group $u_{k+1} = u_+, u_{k+2} = u_-, \ldots, u_{2k} = u_-$. The combinatorial factor summing over all such choices is k^2. The differential equation for this degenerate case is similar to (152), but the combinations of $(u_2 - u_j)$ for $j = k+2, \ldots, 2k$, and $(u_i - u_{k+1})$ for $i = 1, 3, \ldots, k$ are eliminated. One then modifies (151) to be

$$f = \sqrt{|(u_1 - u_2)(u_3 - u_4)(u_1 - u_3)(u_2 - u_4)|}\, g \tag{203}$$

for this purpose, in the case $k = 2$ given as an example. Then g satisfies a differential equation which yields the terms proportional to e^{2ix} as a power series in $1/x$ up to order $1/x^3$. The first order is

$$g^{(1)} = \frac{k^2[(k-1)^2 - 1]}{4x} \tag{204}$$

and it vanishes in the zero-replica limit. At second order, one has

$$g^{(2)} = \frac{9k^2[(k-1)^2 + 1]}{32x^2} + \frac{3k^2(k-1)^2(k-2)}{16x^2} + \frac{k^2[(k-1)^2(k-2)^2 - 2(k-1)^2]}{32x^2} \tag{205}$$

which come from the contribution of three different diagrams connecting two lines; a double line, a connected line, a separate two lines. The diagrams are the same as for the non-oscillating calculation, but the weight factors are different. In the zero-replica limit, it gives $g^{(2)} = k^2/4x^2$. At third order, we have seven different diagrams. Adding their contributions, one finds in the zero-replica limit

$$g^{(3)} = -\frac{k^2}{2x^3}. \tag{206}$$

If one compares this series with the terms proportional to e^{2ix} in (201),

$$\frac{k^2}{4} e^{2ix} \left(\sum_p \frac{(p-1)}{2x^p} (-i)^{p-1} \right)^2 = \frac{k^2}{4} e^{2ix} \left[\frac{1}{4x^2} - \frac{i}{2x^3} + \cdots \right] \tag{207}$$

one sees that the two agree up to this order.

We have a term $\frac{e^{ikx}}{x^{k^2}}$. By the second derivative of this term, we also have a k^2 term as

$$\frac{d^2}{dx^2} \frac{e^{ikx}}{x^{k^2}} = -k^2 \left(1 - \frac{1}{x^2} \right) + O(k^3). \tag{208}$$

We also have similar terms from the derivative of $\frac{e^{-ikx+2ix}}{x^{k^2}}$. Together we have $1 - 2\left(\frac{\sin x}{x}\right)^2$. Adding this term to (199), we obtain the two-point correlation function of GOE,

$$\rho_2(\lambda, \mu) = \rho^2(\lambda) \left[1 - \left(\frac{\sin x}{x} \right)^2 - \frac{d}{dx} \frac{\sin x}{x} \int_x^\infty \frac{\sin z}{z}\, dz \right]. \tag{209}$$

Thus we have shown here that replica limit of the moment of the characteristic polynomial gives the consistent result with the well-known resolvent two-point correlation functions both for GUE and GOE.

10. Gaussian symplectic ensemble

The Gaussian symplectic ensemble (GSE) is easily formulated as an extension of the GOE. Let X be a quaternion symmetric matrix. Let us consider as an example the $N = 2$ case; X_{11} and X_{22} are both real numbers, whereas the off-diagonal element X_{12} is a quaternion and X_{21} its conjugate. The quaternion X_{12} may be written as

$$X_{12} = a + be_1 + ce_2 + de_3 \tag{210}$$

where e_i are the quaternion basis (i.e. up to a relabelling and a factor i, the Pauli matrices); in the basis $e_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$, the coefficients a, b, c and d are real numbers. One can write instead the matrix X as ordinary 4×4 matrix X', the elements are then the usual complex numbers,

$$X' = \begin{pmatrix} x_{11} & 0 & u & \mathrm{i}v \\ 0 & x_{11} & \mathrm{i}v^* & u^* \\ u^* & -\mathrm{i}v & x_{22} & 0 \\ -\mathrm{i}v^* & u & 0 & x_{22} \end{pmatrix} \tag{211}$$

where x_{11} and x_{22} are real, u and v complex. The relation $[\det(\lambda - X)]^2 = \det(\lambda - X')$ allows us to write

$$F_2(\lambda, \mu) = \left\langle \frac{[\det(\lambda - X)]^2}{[\det(\mu - X)]^2} \right\rangle = \left\langle \frac{\det(\lambda - X')}{\det(\mu - X')} \right\rangle. \tag{212}$$

The density of state $\rho(\lambda)$ may be easily deduced from F_2. From the relation (212), one obtains in the $N = 2$ case,

$$F_2(\lambda, \mu) = \int \oint \frac{(\lambda - t_1)^2 (\lambda - t_2)^2}{(\mu - t_1)^2 (\mu - t_2)^2} (t_1 - t_2)^4 \, \mathrm{e}^{-(t_1^2 + t_2^2)} \frac{\mathrm{d}t_1}{2\pi} \frac{\mathrm{d}t_2}{2\pi} \tag{213}$$

in which t_1 and t_2 are the eigenvalues of X. The imaginary part of F_2, from which one deduces the density of eigenvalues through

$$\rho(\lambda) = \frac{1}{\pi N} \lim_{\mu \to \lambda} \mathrm{Im} \frac{\partial}{\partial \lambda} F_2(\lambda, \mu) \tag{214}$$

is obtained by picking up the contribution of the pole $t_1 = \mu$. Taking the imaginary part, a derivative with respect to λ and setting $\mu \to \lambda$ afterwards, one obtains

$$\rho(\lambda) = \int (\lambda - t_2)^4 \, \mathrm{e}^{-(\lambda^2 + t_2^2)} \frac{\mathrm{d}t_2}{2\pi} \tag{215}$$

which may easily be checked directly.

For the inverse of the characteristic polynomial, we find easily the following formula in the X' representation (still for the $N = 2$ example),

$$\begin{aligned} \left\langle \frac{1}{\det(\mu - X')} \right\rangle &= \int \prod_{i=1}^{4} \mathrm{d}z_i \, \mathrm{d}z_i^* \, \mathrm{d}X' \, \mathrm{e}^{\mathrm{i}2 z_a^* (\mu \delta_{ab} - X'_{ab}) z_b - \mathrm{tr} X'^2} \\ &= \int \prod_{i=1}^{4} \mathrm{d}z_i \, \mathrm{d}z_i^* \, \mathrm{e}^{-2(\sum z_i^* z_i)^2 + \mathrm{i}2\mu z_a^* z_a} \\ &= \frac{\pi^2}{4} \int \mathrm{d}b \prod_{i=1}^{4} \mathrm{d}z_i \, \mathrm{d}z_i^* \, \mathrm{e}^{-2b^2 - 2\mathrm{i}b \sum z_i^* z_i + 2\mathrm{i} \sum \mu z_i^* z_i} \\ &= \int \mathrm{d}b \frac{1}{(\mu - b)^4} \, \mathrm{e}^{-2b^2}. \end{aligned} \tag{216}$$

The imaginary part is the Hermite polynomial $H_3(\mu)$.

For general N, similarly, we find that the expectation value of the inverse of the characteristic polynomial is

$$\left\langle \frac{1}{\det(\mu - X')} \right\rangle = \int \mathrm{d}b \frac{1}{(\mu - b)^{2N}} \,\mathrm{e}^{-Nb^2} \tag{217}$$

and its imaginary part is simply $H_{2N-1}(\mu)$.

For the inverse of the product of two characteristic polynomials, one obtains

$$\left\langle \frac{1}{(\det(\mu_1 - X))^2 (\det(\mu_2 - X))^2} \right\rangle = \int \frac{1}{[\det(B)]^{2N}} \,\mathrm{e}^{-\frac{N}{2}\mathrm{tr} B^2 + N \mathrm{tr} BM} \,\mathrm{d}B \tag{218}$$

where X is an $N \times N$ quaternion symmetric matrix, B a 2×2 quaternion matrix and M is a diagonal matrix $M = \mathrm{diag}(\mu_1, \mu_2)$.

The average of the square of the characteristic polynomials may then be written as

$$\langle [\det(\lambda - X)]^2 \rangle = \int \mathrm{d}A [\det(\lambda - A)]^N \,\mathrm{e}^{-\frac{N}{2}\mathrm{tr} A^2} \tag{219}$$

where A is a 2×2 real symmetrix matrix. The quantity $[\det(\lambda - A)]^N$ is a polynomial in λ of order λ^{2N}.

Finally, the ratio $F_N(\lambda, \mu)$ may be written as an integral over a supermatrix Q, the derivation being similar to that for GOE,

$$F_N(\lambda, \mu) = \int \frac{1}{(\mathrm{Sdet}\, Q)^N} \,\mathrm{e}^{-N \mathrm{Str}\, Q^2 + \mathrm{i} N \mathrm{Str}\, Q\Lambda} \,\mathrm{d}Q. \tag{220}$$

A supergroup diagonalization and the Itzykson–Zuber integral (for which we may use the same formulae as for GOE since the Jacobian has the same form after the diagonalization) lead then to

$$F_N(\lambda, \mu) = \int \mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathrm{d}u \frac{(t_1 t_2)^N}{u^{2N}} \frac{|t_1 - t_2| (\lambda - \mu)^2}{(u - \mathrm{i}t_1)(u - \mathrm{i}t_2)} \left[1 - \frac{1}{N(\lambda - \mu)} \left(\frac{1}{u - \mathrm{i}t_1} + \frac{1}{u - \mathrm{i}t_2} \right) \right] + \delta_{\lambda,\mu}. \tag{221}$$

The density of states $\rho(\lambda)$ follows:

$$\rho(\lambda) = \left\langle \frac{1}{N} \mathrm{Tr}\, \delta(\lambda - X) \right\rangle = \lim_{\mu \to \lambda} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathrm{d}u \frac{(t_1 t_2)^N}{u^{2N}} \frac{|t_1 - t_2| (2u - \mathrm{i}t_1 - \mathrm{i}t_2)}{(u - \mathrm{i}t_1)^2 (u - \mathrm{i}t_2)^2}$$
$$\times \,\mathrm{e}^{-\frac{N}{2}(t_1^2 + t_2^2) - N u^2 - \mathrm{i} N (t_1 + t_2)\lambda - 2\mathrm{i} N u \mu}.$$

The formula is quite similar to that of the GOE, except that the combination $\frac{(t_1 t_2)}{b^2}$ is raised here to the power $2N$ instead of $-N$. This difference makes the calculation of the imaginary part easier, since the contour integral on b gives a contribution to the imaginary part, similar to that of GUE, and therefore the result does not involve the incomplete Gaussian integrals (such as $B(x)$ in (59)), which appear in the GOE case.

Let us consider now a ratio of characteristic polynomials $F_N(\lambda_1, \lambda_2, \mu_1, \mu_2)$ defined as

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \left\langle \frac{[\det(\lambda_1 - X)\det(\lambda_2 - X)]^2}{[\det(\mu_1 - X)\det(\mu_2 - X)]^2} \right\rangle. \tag{222}$$

With the supermatrix formalism and its supergroup diagonalization, one can write again an integral over eigenvalues, similar to (120) in the GOE case,

$$F_N(\lambda_1, \lambda_2, \mu_1, \mu_2) = \int \frac{(t_1 t_2 t_3 t_4)^N}{(u_1 u_2)^{2N}} \frac{\prod_{i<j}^4 |t_i - t_j| (u_1 - u_2)^4}{\prod_{\alpha=1}^2 \prod_{k=1}^4 (u_\alpha - \mathrm{i}t_k)} \,\mathrm{e}^{-\frac{N}{2}\sum t_i^2 - N \sum u_\alpha^2}$$
$$\times I \,\mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathrm{d}t_3 \,\mathrm{d}t_4 \,\mathrm{d}u_1 \,\mathrm{d}u_2 \tag{223}$$

where I is the HIZ integral,

$$I = \int \mathrm{d}g\, \mathrm{e}^{N \mathrm{Str}\, g \hat{Q} g^{-1} \Lambda} \tag{224}$$

with $\hat{Q} = \mathrm{diag}(\mathrm{i}t_1, \mathrm{i}t_2, \mathrm{i}t_3, \mathrm{i}t_4, u_1, u_1, u_2, u_2)$, and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \mu_1, \mu_1, \mu_2, \mu_2)$.

The HIZ integral is in fact identical to that for the GOE case with corresponding variables. In the supergroup integration for the GOE case one found a product of two series, one infinite, and the other one finishing after a finite number of terms. The same structure appears in the present supergroup integration for the GSE case, except that the finite and infinite series have their respective variables switched.

Thus in the large-N limit, one has to consider, as in section 8, three types of saddle points; the saddle points of classes (i) and (ii) give the same answer. The saddle points for t_i and u_α become

$$t_\pm = \frac{-\mathrm{i}\lambda \pm \sqrt{4-\lambda^2}}{2} \qquad u_\pm = \frac{\mu \pm \mathrm{i}\sqrt{4-\mu^2}}{2}. \tag{225}$$

Note that for the present GSE case, the density of state $\rho(\lambda)$ is given by $\rho(\lambda) = \sqrt{4-\lambda^2}/2\pi$. Therefore, by the same arguments as for the GOE, we obtain the two-point correlation function

$$\rho_2(\lambda_1, \lambda_2) = \rho^2(\lambda) \left[1 - \left(\frac{\sin 2x}{2x} \right)^2 - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sin 2x}{2x} \right) \int_x^\infty \frac{\sin 2z}{2z}\, \mathrm{d}z \right]. \tag{226}$$

We have neglected the third class of the saddle point (iii) for the GOE. In the GSE, those saddle points do contribute, since the imaginary part is taken from a contour integral over b_1 and b_2. Since the HIZ formula for $\beta = 4$, $k = 1$ takes a simple form, we obtain for the saddle point values of those b $\left(b_1^+, b_2^-\right)$, and thus

$$I = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin 2x}{2x}. \tag{227}$$

For the corresponding t integral, we take the following saddle points: a set of saddle points such as $\left(t_1^+, t_2^-, t_3^+, t_4^-\right)$ gives the non-oscillating constant contribution. Note that $t_1^+ t_2^- = 1$, and $\mathrm{i}N\left(t_1^+ + t_2^-\right)\lambda_1 = -\lambda^2 N$. Thus there is a correction to (226) by (227). By taking the normalized coefficient, which makes the two-point correlation function become $O(x^4)$ in the small x limit, a property of the GSE, we find the scaling limit of the two-point correlation function,

$$\begin{aligned} \rho_2(\lambda_1, \lambda_2) &= \rho^2(\lambda) \left(1 - \left(\frac{\sin 2x}{2x} \right)^2 - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\sin 2x}{2x} \right) \left[\int_x^\infty \frac{\sin 2z}{2z}\, \mathrm{d}z - \frac{\pi}{2} \right] \right) \\ &= \rho^2(\lambda) \left(1 - \left(\frac{\sin 2x}{2x} \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin 2x}{2x} \right) \int_0^x \frac{\sin 2z}{2z}\, \mathrm{d}z \right). \end{aligned} \tag{228}$$

11. Extension to an external matrix source

Let us consider an external matrix source A be coupled to a real symmetric random matrix, or to a quaternion self-dual random matrix. The corresponding Gaussian probability measure

$$P_A(X) = \frac{1}{Z} \epsilon^{-\frac{N}{2} \mathrm{tr} X^2 + N \mathrm{tr} AX} \tag{229}$$

has lost the invariance under GOE or GSE.

In a previous article [7], we have considered this external source problem for the correlation functions $\langle \prod_{\alpha=1}^{k} \det(\lambda_\alpha - X)\rangle$, in which X is a real symmetric random matrix. This is done by integrating over X

$$\int \mathrm{e}^{-\frac{N}{2}\mathrm{tr}X^2+N\mathrm{tr}AX+\mathrm{i}N\mathrm{tr}XY}\,\mathrm{d}X = \mathrm{e}^{-\frac{N}{4}\mathrm{tr}[(Y-\mathrm{i}A)^2+(Y-\mathrm{i}A)(Y^T-\mathrm{i}A)]} \tag{230}$$

where $Y = -\sum_{\alpha=1}^{k}\bar{\theta}_{\alpha a}\theta_{\alpha b}$. One may assume, without loss of generality, that A is a diagonal matrix. Then, the only new term with respect to the zero-source case is $\exp\left[-\mathrm{i}N\sum a_j\bar{\theta}_{\alpha j}\theta_{\alpha j}\right]$. This diagonal term modifies the previous determinant $(\det B)^N$, and gives instead $\prod_{j=1}^{N}\prod_{l=1}^{k}(t_l - \mathrm{i}a_j)$. In this way we had obtained in [4] that, when all the λ_j are equal to λ, one has

$$\langle[\det(\lambda - X)]^k\rangle = \mathrm{e}^{-N\sum\lambda_l^2}\int\prod_{l=1}^{k}\prod_{j=1}^{N}(t_l - \mathrm{i}a_j)\prod_{l<l'}(t_l - t_{l'})^4\,\mathrm{e}^{-N\sum t_l^2+2\mathrm{i}N\lambda\sum t_l}\prod_{l=1}^{k}\mathrm{d}t_l. \tag{231}$$

Similarly for the ratio of two characteristic polynomials, we have $Y = -\sum_{\alpha=1}^{k}\bar{\theta}_{\alpha a}\theta_{\alpha b} - z_{\alpha a}^{*}z_{\alpha b}$ as in (45). The external source gives a diagonal shift for θ and z. This leads to the modification of the Sdet term. For instance, in the $k = 1$ case, one replaces the super-determinant $[\mathrm{Sdet}]^{\frac{N}{2}}$ by

$$\frac{1}{[\mathrm{Sdet}Q]^{\frac{N}{2}}} \to \prod_{\gamma=1}^{N}\frac{(a_\gamma - \mathrm{i}t)}{[(a_\gamma - u_1)(a_\gamma - u_2)]^{\frac{1}{2}}}. \tag{232}$$

Then the density of states, for the GOE modified by an external source matrix, is given by

$$\rho(\lambda) = -\frac{1}{8\pi^2 N}\mathrm{Im}\lim_{\mu\to\lambda}\int \mathrm{d}t\,\mathrm{d}u\prod_{\gamma=1}^{N}\frac{(a_\gamma - \mathrm{i}t)}{[(a_\gamma - u_1)(a_\gamma - u_2)]^{\frac{1}{2}}}\frac{|u_1 - u_2|}{(\mathrm{i}t - u_1)(\mathrm{i}t - u_2)}$$
$$\times\left[\frac{1}{\mathrm{i}t - u_1} + \frac{1}{\mathrm{i}t - u_2}\right]\mathrm{e}^{-2\mathrm{i}Nt\lambda+N(u_1+u_2)\mu-Nt^2-\frac{N}{2}(u_1^2+u_2^2)}. \tag{233}$$

Let us consider the one-point Green function $G(z)$ (with an external source),

$$G(z) = \left\langle \mathrm{tr}\frac{1}{z - X}\right\rangle \tag{234}$$

for a real symmetric random matrix X. It is given by the derivative of $F(\lambda, \mu)$,

$$G(z) = \lim_{\lambda=\mu=z}\frac{\partial}{\partial\lambda}\left\langle\frac{\det(\lambda - X)}{\det(\mu - X)}\right\rangle$$
$$= \int\prod_{\gamma=1}^{N}\frac{a_\gamma - \mathrm{i}t}{(u_1 - a_\gamma)(u_2 - a_\gamma)}\frac{|u_1 - u_2|(2\mathrm{i}t - u_1 - u_2)}{(\mathrm{i}t - u_1)^2(\mathrm{i}t - u_2)^2}$$
$$\times\,\mathrm{e}^{-2\mathrm{i}Nt\lambda+N(u_1+u_2)\mu-Nt^2-\frac{N}{2}(u_1^2+u_2^2)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2. \tag{235}$$

Using the change of variables, $b = \frac{u_1+u_2}{2}$, $r = \frac{(u_1-u_2)^2}{4}$, we obtain

$$G(z) = \int\mathrm{d}t\int\mathrm{d}b\int_0^\infty\mathrm{d}r\frac{(\mathrm{i}t - b)}{((\mathrm{i}t)^2 - 2\mathrm{i}tb + b^2 - r)^2}\,\mathrm{e}^{-2\mathrm{i}Nt\lambda+2Nb\mu-Nt^2-N(b^2+r)}$$
$$\times\prod_\gamma\frac{a_\gamma - \mathrm{i}t}{(b^2 - r - 2a_\gamma b + a_\gamma{}^2)^{1/2}}. \tag{236}$$

After integration by parts over r, we obtain an expression, similar to the GUE case,

$$G(z) = \int\prod_\gamma\left(1 - \frac{\mathrm{i}t}{N(b - a_\gamma)}\right)\frac{1}{\mathrm{i}t}\,\mathrm{e}^{-\frac{1}{2N}t^2-\mathrm{i}tb}\frac{\mathrm{d}b}{2\pi\mathrm{i}}\frac{\mathrm{d}t}{2\pi}. \tag{237}$$

In the large-N limit, one may make the replacement

$$\prod_{\gamma=1}^{N}\left(1-\frac{\mathrm{i}t}{N(b-a_\gamma)}\right)\simeq\exp\left(-\frac{\mathrm{i}t}{N}\sum_{\gamma=1}^{N}\frac{1}{b-a_\gamma}\right). \tag{238}$$

Let us denote the (non-random) density of states of the external matrix A as $\rho_0(a)$, we may then write the rhs of (238) as

$$\exp\left(-\mathrm{i}t\int\mathrm{d}a\frac{\rho_0(a)}{u-a}\right). \tag{239}$$

We define the resolvent of the external source $G_0(z)$

$$G_0(z)=\int\mathrm{d}a\frac{\rho_0(a)}{z-a}. \tag{240}$$

From (237) and (238), we obtain

$$\frac{\partial G}{\partial z}=\oint\frac{\mathrm{d}b}{2\pi\mathrm{i}}\frac{1}{u+G_0(u)-z}. \tag{241}$$

The contour surrounds all the eigenvalues a_γ. The zeros of the denominator satisfy

$$u+\frac{1}{N}\sum\frac{1}{u-a_\gamma}=z. \tag{242}$$

As discussed in a previous paper [16], we take the poles of $\hat{u}(z)=z-\frac{1}{z}+O\left(\frac{1}{z^2}\right)$ and $u=\infty$, and then

$$\frac{\partial G}{\partial z}=1-\frac{1}{1+\frac{\mathrm{d}G_0}{\mathrm{d}\hat{u}(z)}}=1-\frac{\mathrm{d}\hat{u}(z)}{\mathrm{d}z}. \tag{243}$$

The integration gives

$$G(z)=z-\hat{u}(z). \tag{244}$$

Since $\hat{u}(z)$ is a solution of $u+G_0(u)=z$, we obtain the following equation, due to Pastur [28], for the GUE

$$G(z)=G_0(z-G(z)). \tag{245}$$

Thus we obtain the same Pastur equation as for the GUE case; this is easily understandable from a diagrammatic analysis; in the large-N limit, planar diagrams are simple rainbow diagrams and do not distinguish between GOE, GSE or GUE at leading order in the large-N limit. For the GSE, a similar algebra would lead to the same equation.

For the two-point correlation function, $k=2$, the same shift for the Sdet occurs in the presence of the external source. We have found in the previous sections the resolvent two-point correlation functions in GOE and GSE in the Dyson scaling limit. We now discuss the two-point correlation function in the Dyson scaling limit when the external source matrix is coupled to the random matrix. We have already given a proof of the universality in GUE. The argument goes as follows for this GUE case [16].

In the presence of a matrix source there is a kernel $K_N(\lambda_1,\lambda_2)$ given by

$$K_N(\lambda_1,\lambda_2)=\int\frac{\mathrm{d}t}{2\pi}\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{\mathrm{i}t}\prod_{\gamma=1}^{N}\left(1+\frac{\mathrm{i}t}{N(u-a_\gamma)}\right)\mathrm{e}^{-\frac{t^2}{2N}-\mathrm{i}ut-\mathrm{i}t\lambda_1+Nu(\lambda_1-\lambda_2)} \tag{246}$$

from which all the n-point correlation functions may be obtained by the usual determinant formulae of a matrix whose elements are the $K_N(\lambda_i,\lambda_j)$. Defining the scaling variable

$y = N(\lambda_1 - \lambda_2)$, and using, in the large-N limit, the expression of (238) for the product $\prod_{\gamma=1}^{N}\left(1 + \frac{\mathrm{i}t}{N(u-a_\gamma)}\right)$, after integration over t, one obtains

$$\frac{\partial K_N}{\partial \lambda_1} = \frac{1}{\pi}\mathrm{Im}\oint \frac{\mathrm{d}u}{2\pi\,\mathrm{i}}\frac{1}{u + G_0(u) - \lambda_1 + \mathrm{i}\epsilon}\,\mathrm{e}^{-uy}. \tag{247}$$

Again one defines the pole $\hat{u}$, and obtains

$$\begin{aligned}\frac{\partial K_N}{\partial \lambda_1} &= \frac{1}{\pi}\,\mathrm{Im}\,\frac{\mathrm{d}\hat{u}}{\mathrm{d}\lambda_1}\,\mathrm{e}^{-y\hat{u}(\lambda_1-\mathrm{i}\epsilon)}\\ &= -\frac{1}{\pi y}\frac{\partial}{\partial \lambda_1}\,\mathrm{Im}(\mathrm{e}^{-y\hat{u}(\lambda_1-\mathrm{i}\epsilon)})\end{aligned} \tag{248}$$

in which

$$\hat{u}(\lambda_1 - \mathrm{i}\epsilon) = \lambda_1 - \mathrm{Re}\,G(\lambda_1) - \mathrm{i}\pi\rho(\lambda_1). \tag{249}$$

Therefore one ends up with

$$K_N(\lambda_1, \lambda_2) = -\frac{1}{\pi y}\,\mathrm{e}^{-y[\lambda_1 - \mathrm{Re}G(\lambda_1)]}\sin[\pi y\rho(\lambda_1)]. \tag{250}$$

Putting this expression of the kernel into the correlation functions, one sees that their only dependence in the eigenvalues a_γ of the external source is a scale factor through the density of states; for instance, the connected two-point function is simply

$$\rho_c(\lambda_1, \lambda_2) = -\frac{1}{\pi^2 y^2}\sin^2\left[\pi\rho\left(\frac{\lambda_1+\lambda_2}{2}\right)y\right]. \tag{251}$$

We now consider the universality in the GOE and GSE ensembles in the presence of an external source. In the GOE case, we used a generalized HIZ integral which has two parts. One part is the t-integration, which gives simply

$$g_t = 1 - \frac{\mathrm{i}}{N(t_1 - t_2)(\lambda_1 - \lambda_2)}. \tag{252}$$

This leads now to the integral

$$\int \prod_{\gamma}^{N}(a_\gamma - \mathrm{i}t_1)(a_\gamma - \mathrm{i}t_2)\frac{g_t}{\lambda_1 - \lambda_2}\,\mathrm{e}^{-Nt_1^2 - Nt_2^2 + 2\mathrm{i}Nt_1\lambda_1 + 2\mathrm{i}Nt_2\lambda_2}\frac{\mathrm{d}t_1}{2\pi}\frac{\mathrm{d}t_2}{2\pi}. \tag{253}$$

In the large-N limit, the saddle-point equations for the t_j $(j = 1, 2)$ are

$$t_j - \mathrm{i}\lambda_1 - \frac{1}{2N}\sum_{\gamma}\frac{1}{a_\gamma - \mathrm{i}t_j} = 0. \tag{254}$$

As in (242), there is a saddle point $\hat{t}_j$, which behaves as $\hat{t}_j \simeq \mathrm{i}\lambda_j$ in the large λ_j domain. Then $\hat{t}_j$ becomes

$$\hat{t}_j = \mathrm{i}\lambda_j - \mathrm{i}\,\mathrm{Re}\,G(\lambda_j) + \pi\rho(\lambda_j). \tag{255}$$

Thus, this t-integration in the large-N limit behaves as in the sourceless case, and yields $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)$ where x is $N\pi\rho[(\lambda_1 + \lambda_2)/2](\lambda_1 - \lambda_2)$. More generally, this part may be written as

$$I = \frac{\mathrm{d}}{\mathrm{d}x}K(x, y) \tag{256}$$

where $K(x, y)$ is a kernel. This is due to the HIZ integral for $\beta = 4$. In our case, this kernel is a sine-kernel.

For the u-integration, one has a product

$$\prod_{\gamma}^{N}\prod_{j=1}^{4}\frac{1}{u_j-a_\gamma}=\exp\left[-\sum_{\gamma}\sum_{j}\ln(u_j-a_\gamma)\right]. \tag{257}$$

There is a saddle point $\hat{u}_j$, which behaves as $\hat{u}_1=\hat{u}_2\simeq\lambda_1$ and $\hat{u}_3=\hat{u}_4\simeq\lambda_2$. Using these saddle points, the factor coming from this u-integral is the same as in (179). If we denote this u-integral by I, it is easy to find that the derivative of I with respect to x is the sine-kernel. More generally, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}I=K(x,y). \tag{258}$$

This is due to the HIZ formula for $\beta=1$ which gives

$$\sum\frac{n!}{(\mathrm{i}x)^{n+1}}. \tag{259}$$

In the sine-kernel case, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{\mathrm{i}x}\left(\frac{1}{x}-\frac{\mathrm{i}}{x^2}-\frac{2}{x^3}+\cdots\right)\right]=\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}x}}{x} \tag{260}$$

where the successive cancellation occurs in the higher order.

Thus we have a universal two-point correlation function in the presence of the external source,

$$\rho(\lambda_1,\lambda_2)=\rho^2(\lambda)\left[1-\left(\frac{\sin x}{x}\right)^2-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)\int_x^\infty\frac{\sin z}{z}\,\mathrm{d}z\right] \tag{261}$$

where $x=\pi N\rho(\lambda)(\lambda_1-\lambda_2)$ and the external source a_γ appears only in the density of states ρ. The GSE case may be analysed as the GOE and also yields a universal correlation function with respect to the external source eigenvalues, as in (228).

12. Universalities at the edges in GOE

Near edges of the support of the density of states, it is well known that a new scaling behaviour takes place [29, 31]. In the simplest case of the Wigner semi-circle the behaviour is governed by an Airy kernel. Let us consider now the edge behaviour of the two-point correlation of characteristic polynomials,

$$F_N(\lambda_1,\lambda_2)=\langle\det(\lambda_1-X)\det(\lambda_2-X)\rangle \tag{262}$$

when X is a real symmetric random matrix. If λ_1 and λ_2 are within the bulk of the support of the asymptotic density of states, we have found in the Dyson scaling limit that [7]

$$F_N(\lambda_1,\lambda_2)=\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}. \tag{263}$$

As we have discussed earlier, by using Grassmann variables, one finds for arbitrary λ_1, λ_2 and finite N,

$$\begin{aligned}F_N(\lambda_1,\lambda_2)&=\int(\det B)^N\,\mathrm{e}^{-N\mathrm{tr}B^2+\mathrm{i}N\mathrm{tr}B\Lambda}\\&=\int(t_1t_2)^N\,\mathrm{e}^{-Nt_1^2-Nt_2^2-2\mathrm{i}Nt_1\lambda_1-2\mathrm{i}Nt_2\lambda_2}\left[\frac{(t_1-t_2)^2}{(\lambda_1-\lambda_2)^2}-\frac{\mathrm{i}(t_1-t_2)}{(\lambda_1-\lambda_2)^3}\right]\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned} \tag{264}$$

In the large-N limit, the saddle points for the t_a are

$$t_a=\frac{-\mathrm{i}\lambda_a\pm\sqrt{2-\lambda_a^2}}{2}. \tag{265}$$

The critical point corresponds to a degenerate quadratic form of fluctuations near the saddle point; expanding then to the next order (since there is a flat direction), one finds a new scaling limit when the parameters α and β are $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$, when the λ are at distance $N^{-\alpha}$ of the end point $\sqrt{2}$, and the t_a at distance $N^{-\beta}$ of $-\mathrm{i}/\sqrt{2}$. Performing the scaling change of variables

$$\begin{aligned}\lambda_a &= \sqrt{2} - N^{-\alpha} x_a \\ t_a &= -\frac{\mathrm{i}}{\sqrt{2}} + N^{-\beta}\tau_a\end{aligned} \tag{266}$$

the integral becomes, in this regime of large N,

$$\begin{aligned}F^{(1)}(\lambda_1,\lambda_2) &= \int \mathrm{e}^{-\frac{2\sqrt{2}}{3}\mathrm{i}(\tau_1^3+\tau_2^3)+2\mathrm{i}(x_1\tau_1+x_2\tau_2)}\left[\frac{(\tau_1-\tau_2)^2}{(\lambda_1-\lambda_2)^2}+\frac{(\tau_1-\tau_2)}{(\lambda_1-\lambda_2)^3}\right]\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &= \frac{1}{(\lambda_1-\lambda_2)^2}[Ai''(x_1)Ai(x_2) - 2Ai'(x_1)Ai'(x_2) + Ai(x_1)Ai''(x_2)]\end{aligned} \tag{267}$$

in which use has been made of the differential equation satisfied by the Airy function, $Ai''(x) = xAi(x)$. Noting that

$$\begin{aligned}&\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)[Ai'(x_1)Ai(x_2) - Ai'(x_2)Ai(x_1)] \\ &\quad = -[Ai''(x_1)Ai(x_2) - 2Ai'(x_1)Ai'(x_2) + Ai(x_1)Ai''(x_2)]\end{aligned} \tag{268}$$

the second term of (264) is then simply the Airy kernel divided by $(\lambda_1 - \lambda_2)$. One ends up with

$$F_N(\lambda_1,\lambda_2) = \frac{1}{x_1-x_2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\left[\frac{Ai'(x_1)Ai(x_2) - Ai'(x_2)Ai(x_1)}{x_1-x_2}\right]. \tag{269}$$

This confirms the result (256), which was stated for more general kernels.

Extending this analysis to ratios of characteristic polynomials, one obtains similarly another term, expressible as an integral of the kernel. Thus we have obtained the two-point correlation function near the edge for the GOE ensemble. The same argument can easily be transposed to the GSE.

Let us now consider other edge problems that one meets when an external source matrix is added to the probability distribution. For instance, one can tune the external source to create a gap in the spectral density of states $\rho(\lambda)$. At the critical point at which this gap closes, a new universality class appears and we have studied earlier the new scaling behaviour at the origin [24, 25].

It was found that the kernel $K_N(x, y)$ for the GUE ensemble in the appropriate scaling limit was

$$K(x,y) = \frac{\hat{\phi}'(x)\hat{\psi}'(y) - \hat{\phi}''(x)\hat{\psi}(y) - \hat{\phi}(x)\hat{\psi}''(y)}{x-y}. \tag{270}$$

For the GOE case, in the critical domain of the gap closing point, one finds a two-point correlation function,

$$\rho(x,y) = 1 - K(x,y)K(y,x) - \frac{\mathrm{d}}{\mathrm{d}x}K(x,y)\int_x^\infty K(z,y)\,\mathrm{d}z. \tag{271}$$

13. Level spacing distribution in GOE and GSE

The level spacing probability function $E(s)$, the probability that there is no eigenvalue inside the interval $[-s/2, s/2]$, is given by the Fredholm determinant

$$E(s) = \det[1 - \hat{K}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\frac{s}{2}}^{\frac{s}{2}} \cdots \int_{-\frac{s}{2}}^{\frac{s}{2}} \det[K(x_i, x_j)]_{i,j=1,\ldots,n} \prod_{k=1}^{n} \mathrm{d}x_k. \tag{272}$$

Let us briefly review the derivation of this formula for $E(s)$ [30, 25]. This level spacing function is expressed by an Hamiltonian formalism. The derivation is due to Tracy and Widom [30].

In the GUE case, we have

$$E(a, b) = \det[1 - \hat{K}] \tag{273}$$

where we work with a general interval (a, b), and later specialize it to $(a, b) = \left(-\frac{s}{2}, \frac{s}{2}\right)$. The kernel $K(x, y)$ acts on this interval, and we define

$$\hat{K}(x, y) = K(x, y)\theta(y - a)\theta(b - y) \tag{274}$$

in which $\theta(x)$ is the Heaviside function. By definition, the derivative of E with respect to the end points is

$$\frac{\partial \ln E(a, b)}{\partial b} = -\tilde{K}(b, b) \tag{275}$$

where the Fredholm resolvant $\tilde{K}$ is

$$\tilde{K} = \frac{\hat{K}}{1 - \hat{K}}. \tag{276}$$

The derivative with respect to a is similar. Then, one obtains

$$\frac{\mathrm{d}E(s)}{\mathrm{d}s} = \frac{1}{2}\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial a}\right) \ln E(s)_{b=-a=s/2} = -\tilde{K}\left(\frac{s}{2}, \frac{s}{2}\right). \tag{277}$$

This leads to

$$E(s) = \exp\left[-\int_0^s H\left(\frac{z}{2}, \frac{z}{2}\right) \mathrm{d}z\right] \tag{278}$$

where we write $H(x, x) = \tilde{K}(x, x)$. For GUE, the kernel is the sine-kernel,

$$K(x, y) = \frac{\phi(x)\phi'(y) - \phi'(x)\phi(y)}{x - y}. \tag{279}$$

Operating with K on ϕ, one obtains

$$q(x) = \langle x| \frac{1}{1 - \hat{K}} |\phi\rangle \qquad p(x) = \langle \phi'| \frac{1}{1 - \hat{K}} |x\rangle. \tag{280}$$

This leads to

$$\tilde{K}(x, y) = \frac{q(x)p(y) - q(y)p(x)}{x - y}. \tag{281}$$

Taking the derivatives of $q(x)$ and $p(x)$ with respect to b, one obtains a set of equations which constitute the Hamiltonian system,

$$\dot{Q} = P\left(1 - \frac{2Q^2}{b}\right) \qquad \dot{P} = Q\left(\frac{2P^2}{b} - 1\right) \tag{282}$$

where $Q(b) = q(b, -b : b)$ and $P(b) = p(b, -b : b), q(b, -b : b)$, obtained by setting $a = -b, x = b$ in $q(b, a : x)$. (We have written explicitly the interval dependence of $q(x)$.) The Hamiltonian H which governs this dynamical system is simply

$$H(b, b) = P^2 + Q^2 - \frac{2P^2Q^2}{b} \tag{283}$$

(P and Q have Hamiltonian form, $\dot{Q} = \frac{\partial H}{\partial P}$, $\dot{P} = -\frac{\partial H}{\partial Q}$, with $H = \tilde{K}(b, b)$). For small s, we obtain from these two equations,

$$P = 1 + s + \frac{7}{8}s^2 + \frac{65}{72}s^3 + \cdots \tag{284}$$

$$Q = \frac{s}{2} - \frac{s^3}{48} + \cdots. \tag{285}$$

This leads to

$$E(s) = 1 - s + O(s^4) \tag{286}$$

which agrees with the well-known result [3].

For the GOE ensemble, the kernel is a quaternion matrix, and the correlation functions are expressed by a quaternion determinant.

The matrix kernel of GOE, $\sigma(x, y)$, is

$$\sigma(x, y) = \begin{pmatrix} s(x, y) & Ds(x, y) \\ Js(x, y) & s(x, y) \end{pmatrix} \tag{287}$$

where we denote the sine-kernel as $s(x, y) = \sin(x - y)/(x - y)$, and

$$Js(x) = \int_0^x s(y)\,\mathrm{d}y - \epsilon(x) = \epsilon s(x) - \epsilon \tag{288}$$

where $\epsilon(x) = \frac{1}{2}\mathrm{sgn}\, x$.

Tracy and Widom have found the Fredholm determinant for this matrix kernel [30]. Their result for the interval (a, b) reads

$$E(s) = \exp\left[-\frac{1}{2}\int_0^s H(z, z)\,\mathrm{d}z - \frac{1}{2}\int_0^s H(z, -z)\,\mathrm{d}z\right] \tag{289}$$

for GOE. The Hamiltonian H is the same as the Hamiltonian for the GUE given in (283). We have

$$H(b, -b) = \frac{Q(b)P(b)}{b}. \tag{290}$$

For small s, one obtains from (289)

$$E(s) = 1 - s + \tfrac{1}{36}s^3 + O(s^4) \tag{291}$$

which agrees with the known result [3]. (We have dropped a factor π in the sine-kernel. The correct coefficient is $\frac{\pi^2}{36}$.)

For GSE, we have

$$E\left(\frac{s}{2}\right) = \frac{1}{2}\left[\exp\left[-\frac{1}{2}\int_0^s (H(z, z) + H(z, -z))\,\mathrm{d}z\right] + \exp\left[-\frac{1}{2}\int_0^s (H(z, z) - H(z, -z))\,\mathrm{d}z\right]\right]. \tag{292}$$

For small s, it gives

$$E(s) = 1 - s + O(s^4). \tag{293}$$

When the external source matrix A has only two distinct eigenvalues $-a$ and $+a$, each of them $\frac{N}{2}$ times degenerate, a gap in the spectrum around the origin may be created by appropriately tuning the parameter a. At some critical value of a the gap closes and its vicinity leads to a new interesting universality class. For the GUE case, we have given in an earlier work [25] the equation satisfied by $E(s)$ in the appropriate scaling vicinity of the gap closing point. In this gap closing case, the problem is again governed by a Hamiltonian system with now three different Q_i, P_i $(i = 0, 1, 2)$, as shown in [25]. They satisfy

$$\dot{Q}_n = \frac{\partial H}{\partial P_n} \tag{294}$$

$$\dot{P}_n = -\frac{\partial H}{\partial Q_n}. \tag{295}$$

The Hamiltonian $H = \tilde{K}(b, b)$ reads

$$H = bP_2Q_0 + Q_2P_1 + Q_1P_0 - uP_1Q_0 - vP_2Q_1 + \frac{1}{b-a}[P_1(b)Q_1(a) - Q_2(b)P_2(a) - Q_0(b)P_0(a)][P_1(a)Q_1(b) - Q_2(a)P_2(b) - Q_0(a)P_0(b)] \tag{296}$$

for the interval (a, b).

The kernel $\tilde{K}(b, -b)$ is

$$\tilde{K}(b, -b) = \frac{Q_1(b)P_1(-b) - Q_2(b)P_2(-b) - Q_0(b)P_0(-b)}{2b}. \tag{297}$$

Using expressions of (289) and (292), we obtain the level spacing function both for GOE and GSE near the gap closing point. The second derivative of $E(s)$ is the level spacing probability $p(s)$, which is the probability density that two successive eigenvalues lie at distance s. For s small, it is easy to verify that $p(s)$ is linear in s, a characteristic behaviour of GOE. The small s expansion may be computed from the expansions of Q_n, P_n, which are $Q_0(b) = \frac{\sqrt{2}}{4\pi}\Gamma\left(\frac{1}{4}\right) + \frac{\sqrt{2}}{2\pi^{3/2}}b + O(b^2)$, $Q_1(b) = -\frac{\sqrt{2}}{2\pi}\Gamma\left(\frac{3}{4}\right)b + O(b^2)$, $Q_2(b) = -\frac{\sqrt{2}}{2\pi}\Gamma\left(\frac{3}{4}\right) + O(b)$, $P_0(b) = -\frac{1}{3\sqrt{\pi}}b^3 + O(b^4)$, $P_1(b) = -\frac{1}{\sqrt{\pi}} + O(b)$, $P_2(b) = \frac{1}{\sqrt{\pi}}b + O(b^2)$.

Thus we have obtained new results for the level spacing probablity of GOE. This may be important, for instance, in the discussions of universality for the energy spectrum of quantum dots with interactions [23]. Our study of the external source problem may be related to the questions of the distribution of cycles in the permutations with external source [20] in the GOE case, and to the crystal growth in a random environment [21] or to the spin-glass problem [22].

14. Summary

In this paper, we have studied the correlations of the characteristic polynomials of random matrices, which are either real symmetrix (GOE) or quaternionic self-dual (GSE). It was shown that they are universal in the Dyson limit with respect to an external matrix source linearly coupled to the random matrix. As usual the correlation functions are only sensitive to the external source through a scale set by the mean spacing.

For the ratio of characteristic polynomials, we have applied supersymmetric techniques, and obtained as a by-product the resolvent correlation functions of the GOE and GSE ensembles. The method required using group integrals which are no longer of the semiclassical type studied by Harish-Chandra, Itzykson and Zuber. However, we have found that, in the limit required by the Dyson scaling, these integrals can be performed for both the GOE and the GSE. We have also used an alternative method, in which one computes kth powers of characteristic polynomials, and we have again obtained the correlation functions by a replica method, in the zero-replica limit ($k = 0$).

The level spacing probability has been studied for the GOE and the GSE, and we have given an explicit representation for $E(s)$ in the closing gap case, obtained by tuning the external source.

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References

[1] Dyson F J 1970 *Commun. Math. Phys.* **19** 235
[2] Dyson F J 1972 *J. Math. Phys.* **13** 90
[3] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[4] Brézin E and Hikami S 2000 *Commun. Math. Phys.* **214** 111
[5] Brézin E and Hikami S 2000 *Physica* A **279** 333
[6] Brézin E and Hikami S 2000 *Phys. Rev.* E **62** 3558
[7] Brézin E and Hikami S 2001 *Commun. Math. Phys.* **223** 363
[8] Efetov K 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[9] Harish-Chandra 1956 *Proc. Natl Acad. Sci.* **42** 252
[10] Itzykson C and Zuber J-B 1980 *J. Math. Phys.* **21** 411
[11] Guhr T 1991 *J. Math. Phys.* **32** 336
Guhr T 1996 *Ann. Phys., NY* **250** 145
[12] Alfaro J, Medina R and Urrutia L F 1995 *J. Math. Phys.* **36** 3085
[13] Zirnbauer M 1996 *J. Math. Phys.* **37** 4986
[14] Guhr T and Kohler H 2002 *J. Math. Phys.* **43** 2707
[15] Guhr T and Kohler H 2002 *J. Math. Phys.* **43** 2741
[16] Brézin E and Hikami S 1996 *Nucl. Phys.* B **479** 697
[17] Brézin E and Hikami S 1997 *Phys. Rev.* E **56** 264
[18] Brézin E and Hikami S 1997 *Phys. Rev.* E **55** 4067
[19] Brézin E and Hikami S *Commun. Math. Phys.* (*Preprint* math-ph/0208002) at press
[20] Brézin E and Kazakov V 2000 *Commun. Math. Phys.* **214** 233
[21] Gravner J, Tracy C and Widom H 2002 *Commun. Math. Phys.* **229** 433
[22] Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7647
Parisi G and Potters M 1995 *J. Phys. A: Math. Gen.* **28** 5267
[23] Shapiro B 2001 *Phil. Mag.* B **81** 1303
[24] Brézin E and Hikami S 1998 *Phys. Rev.* E **57** 4140
[25] Brézin E and Hikami S 1998 *Phys. Rev.* E **58** 7176
[26] Kamenev A and Mézard M 1999 *Phys. Rev.* B **60** 3944
[27] Gangard D M and Kamenev A 2001 *Nucl. Phys.* B **610** 578
[28] Pastur L A 1972 *Theor. Math. Phys. (USSR)* **10** 67
[29] Bowick M and Brézin E 1991 *Phys. Lett.* B **268** 21
[30] Tracy C A and Widom H 1996 *Commun. Math. Phys.* **177** 727
[31] Tracy C A and Widom H 1994 *Commun. Math. Phys.* **159** 151